Binomial Expansion Problems

Worksheet on Simultaneous Equations

Solving Systems of Equations
8.1 Method of Substitution
8.2 Method of Elimination

Worksheet on Series and Binomials

Quiz on Wednesday 9.1 & 9.5

**Homework:** 8.1, 8.2
Find the coefficient $a$ of the term in the expansion of the binomial.

6. $(2x - 3y)^8$  
   Term: $ax^6y^2$

\[
\begin{align*}
\binom{8}{2} (2x)^6(-3y)^2 \\
(28)(64x^6)(9y^2)
\end{align*}
\]

<table>
<thead>
<tr>
<th>Calculation</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2^6$</td>
<td>64</td>
</tr>
<tr>
<td>$(-3)^2$</td>
<td>9</td>
</tr>
<tr>
<td>$28 \times 64 \times 9$</td>
<td>16128</td>
</tr>
</tbody>
</table>
Example: Find the coefficient of $x^8$ in the expansion of $(x^2 + 2)^{12}$.

\[
\binom{12}{8} (x^2)^4 (2)^8
\]

\[
(495) (x^8)
\]

<table>
<thead>
<tr>
<th>$12 \text{ nCr } 8$</th>
<th>495</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2^8$</td>
<td>256</td>
</tr>
<tr>
<td>$495 \times 256$</td>
<td>126720</td>
</tr>
</tbody>
</table>
First we need to find $r$. Since $(x^2)^{12} - r = x^8$, $r = 8$. So, the coefficient of $x^8$ is $\binom{12}{8} \cdot 2^8 = 126,720$. 
Solve for $x$ and $y$ in each of the following:

1. $3x + 5y = 7$

$3x + 5y = 7$

$-3x - 9y = -15$

Adding the two equations:

$-4y = -8$

$y = 2$

Method of Elimination

$3x + 5y = 7$

$x + 3(2) = 5$

$x + 6 = 5$

$x = -1$

$3(-1) + 5(2) = 7$

$-3 + 10 = 7$

$7 = 7$
Solve for $x$ and $y$ in each of the following:

1. $3x + 5y = 7$

Substitution

$x + 3y = 5 \quad \Rightarrow \quad x = 5 - 3y$

$3(5 - 3y) + 5y = 7$

$15 - 9y + 5y = 7$

$15 - 4y = 7$

$-4y = -8$

$y = 2$

$x = 5 - 3(2)$

$x = 5 - 6 = -1$

Solution

$(-1, 2)$
Solve for $x$ and $y$ in each of the following:

\[ x + 3y = 5 \]

\[ 3x + 5y = 7 \]

\[ y_1 = \frac{5 - x}{3} \]

\[ y_2 = \frac{7 - 3x}{5} \]
\[
\begin{align*}
2x + 3y &= 5 \\
2x + 6y &= 10 \\
\Rightarrow y' &= \frac{5-x}{3} \\
y^2 &= \frac{10-2x}{6} \\
-2x - 6y &= -10 \\
2x + 6y &= 10 \\
\Rightarrow 0 &= 0
\end{align*}
\]

\[
\begin{align*}
\phi(5-x) &= \frac{6}{3} \\
&= 2(5-x) \\
&= \frac{5-x}{3} \\
y' &= \frac{5}{3} - \frac{x}{3}
\end{align*}
\]
3. \(2x + 6y = 7\)

\[
\begin{align*}
\Rightarrow y_1 &= \frac{5-x}{3} \\
\Rightarrow y_2 &= \frac{7-2x}{6} \\
-2x - 6y &= -10 \\
2x + 6y &= 7
\end{align*}
\]

\[0 \neq -3\]
Systems and

Consistent
one solution

Inconsistent
no solution

Consistently

infinite # of solutions

one point of intersection

systems

parallel
Solving Systems of Equations
Solve two or more equations with the same variables simultaneously. These are called systems of equations.
A solution of a system of equations must be a solution of every equation in the system.
First method of solving a system of equations is the **method of substitution**.
This method involves solving one of the equations for one of the variables, substituting for that variable in the other equation, and then solving this new equation.

**Remember to back-substitute to find the value of the other variable.**
How many solutions?
Examples: Solve by the Method of Substitution. Check your answer by using a graphing utility.

1. \[ x + 2y = 1 \]
   \[ 5x - 4y = -23 \]

\[ x = 1 - 2y \]
\[ 5(1 - 2y) - 4y = -23 \]
\[ 5 - 10y - 4y = -23 \]
\[ 5 - 14y = -23 \]
\[ -14y = -28 \]
\[ y = 2 \]

\[ x = 1 - 2(2) \]
\[ x = 1 - 4 \]
\[ x = -3 \]
2. \[ x^2 + y^2 = 25 \]
\[ 2x + y = 10 \]
\[ y = \frac{1}{2} \sqrt{25-x^2} \]
\[ 9 + 16 = 25 \]
\[ 25 + 0 \leq 25 \]
\[ x^2 + (10 - 2x)^2 = 25 \]
\[ x^2 + 100 - 40x + 4x^2 = 25 \]
\[ 5x^2 - 40x + 75 = 0 \]
\[ 5(x^2 - 8x + 15) = 0 \]
\[ \frac{5(x-5)(x-3)}{x} = 0 \]
\[ x = 5 \quad x = 3 \]
Problem 1: A company has a fixed monthly manufacturing cost of $12,000, and it costs $0.95 to produce each unit. The company then sells each unit for $1.25. How many units must be sold before this company breaks even?

C = 12,000 + 0.95x and  
R = 1.25x. For the break-even point, C = R.

\[
1.25x = 12,000 + 0.95x \\
-0.95x = 12,000 \\
0.30x = 12,000 \\
\therefore x = \frac{12,000}{0.3} = 40,000 \text{ units}
\]
\[
\begin{align*}
C' &= 12000 + 0.95x \\
C' &= 1.25x \\
1.25x &= 12000 + 0.95x \\
0.3x &= 12000 \\
x &= 400,000
\end{align*}
\]
Problem 2: On a Saturday night, the manager of a shoe store evaluates the receipts of the previous week’s sales. Two hundred forty pairs of two different types of tennis shoes were sold. One style sold for $66.95 and the other sold for $84.95. The total receipts for the week were $17,652. The cash register was supposed to record the number of each type of shoe sold but it malfunctioned; help the manager determine how many of each type of shoe were sold.

Let $x =$ type 1
$y =$ type 2

$x + y = 240$

$66.95x + 84.95y = 17,652$

$y = 240 - x$

$66.95x + 84.95(240 - x) = 17,652$

$66.95x + 20,388 - 84.95x = 17,652$

$-18.00x = -2736$

$x = 152$ type 1
Problem 3: Choice of Two Jobs - You are offered two different jobs selling college textbooks. TextsNotDot Com offers an annual salary of $25,000 plus a year end bonus of 1% of your total sales. Cheapbooks offers an annual salary of $20,000 plus a year end bonus of 2% of your total sales. Find the annual sales that make Cheapbooks the better offer.
The Method of Elimination

To use the **method of elimination** to solve a system of two linear equations in \( x \) and \( y \), perform the following steps.

1. Obtain coefficients for \( x \) (or \( y \)) that differ only in sign by multiplying all terms of one or both equations by suitably chosen constants.

2. Add the equations to eliminate one variable; solve the resulting equation.

3. Back-substitute the value obtained in Step 2 into either of the original equations and solve for the other variable.

4. Check your solution in both of the original equations.
Solve the following systems of equations.
\[
\begin{align*}
2x - 3y &= 7 \\
5x + 3y &= 0
\end{align*}
\]
- What are the number of different ways that two lines can intersect. Here is a picture of each of the three ways they can intersect.
In the first graph, we get exactly one solution.
In the second, we get no solution.
In the third, we get infinitely many solutions.
If a system of linear equations has at least one solution, then it is called **consistent**.
If it has no solution, then it is called **inconsistent**.
Definitions:
A system of equations is **consistent** if it has at least one solution. (For a linear system, one or an infinite number of solutions)
A system of equations is **inconsistent** if it has no solution.
Examples: Solve by using the Method of Elimination.

3. 
\[ 2r + 4s = 5 \]
\[ 16r + 50s = 55 \]
4. \[ 7x - 6y = -6 \]
\[ -7x + 6y = -4 \]
### Graphical Interpretation of Solutions

For a system of two linear equations in two variables, the number of solutions is one of the following.

<table>
<thead>
<tr>
<th>Number of Solutions</th>
<th>Graphical Interpretation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Exactly one solution</td>
<td>The two lines intersect at one point.</td>
</tr>
<tr>
<td>2. Infinitely many solutions</td>
<td>The two lines are coincident (identical).</td>
</tr>
<tr>
<td>3. No solution</td>
<td>The two lines are parallel.</td>
</tr>
</tbody>
</table>
Problem 1: A man in a boat can row 8 miles downstream in 1 hour. He can row 6 miles upstream in 3 hours. How fast can the man row in still water and what is the rate of current?
\[
\begin{align*}
3r + 3c &= 24 \\
3(r - c) &= 6 \\
3r - 3c &= 6
\end{align*}
\]

\[
6r = 30 \Rightarrow r = 5 \Rightarrow c = 3
\]

The man can row 5 mph in still water. The rate of current is 3 mph.
Problem 2: You have $10,000 to invest in two simple interest funds. One pays 8% and the other 6%. How much should be invested in each so that the total annual interest is $720?
\[
\begin{align*}
\begin{cases}
  x + y &= 10,000 \\
  0.08x + 0.06y &= 720 \\
\end{cases} \quad \Rightarrow \quad 
\begin{cases}
  -8x - 8y &= -80,000 \\
  8x + 6y &= 72,000 \\
\end{cases} \\
  -2y &= -8,000 \\
  y &= 4,000 \Rightarrow x = 6,000
\end{align*}
\]

You should invest $6,000 at 8% and $4,000 at 6%.
The **elimination method** is based on two key steps. First, obtain opposite coefficients on one of the variables. Second, add the two equations, thus eliminating this variable.
This method does not work but on a few nonlinear systems of equations. It is best to use the substitution method on nonlinear systems of equations.
\[ \frac{x}{2} + 3y = 1 \]

\[ x - y = 2 \]
$y = 2x + 1$

$y = x^2 + 4x - 7$

$y = x^2 + 4x - 7$

$y = 2x + 1$

$0 = x^2 + 2x - 8$
The graph shows two equations:

1. \( y = x + 4 \)
2. \( y = -x^2 + 3 \)
The Method of Substitution

1. Solve one of the equations for one variable in terms of the other.
2. Substitute the expression found in Step 1 into the other equation to obtain an equation in one variable.
3. Solve the equation obtained in Step 2.
4. Back-substitute the value obtained in Step 3 into the expression obtained in Step 1 to find the value of the other variable.
5. Check that the solution satisfies each of the original equations.
1. Evaluate: \[ \sum_{k=2}^{6} (2k - 1) \]

<table>
<thead>
<tr>
<th>k</th>
<th>2k - 1</th>
<th>(2k - 1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>(2(2) - 1)</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>(2(3) - 1)</td>
<td>5</td>
</tr>
<tr>
<td>4</td>
<td>(2(4) - 1)</td>
<td>7</td>
</tr>
<tr>
<td>5</td>
<td>(2(5) - 1)</td>
<td>9</td>
</tr>
<tr>
<td>6</td>
<td>(2(6) - 1)</td>
<td>11</td>
</tr>
</tbody>
</table>

\[ \sum \] 35
2. Find the coefficient $a$ for the given term $ax^7$ in the expansion of $(x - 3)^{13}$.

\[
\binom{13}{6} x^7 (-3)^6 = 1,250,964x^7
\]
3. Find all solutions for the following system of equations:

\[
\begin{align*}
\begin{cases}
x + y &= 0 \\
x^3 - 5x - y &= 0
\end{cases}
\end{align*}
\]

\[
\begin{align*}
&y = -x \\
&-(x) \\
&x^3 - 5x + x = 0 \\
&x^3 - 4x = 0 \\
&x(x^2 - 4) = 0 \\
&x(x+2)(x-2) = 0 \\
&x = 0, \ x = -2, \ x = 2
\end{align*}
\]