Return Quiz

3.5 Rational Functions and Asymptotes
   Rational Functions
   Horizontal and Vertical Asymptotes
   Applications
3.6 Graphs of Rational Functions
3.7 Exploring Data Quadratic Models

Worksheet

Homework:  3.5 & 3.6, 3.7
\[ f(x) = 2x^4 - 17x^3 + 35x^2 + 9x - 45 \]

\[ p = \pm 1 \quad \pm 3 \pm 5 \pm 9 \pm 15 \pm 45 \]

\[ q = \pm 1 \pm 2 \]

\[ n = \pm 1 \pm 3 \pm 5 \pm 9 \pm 15 \pm 45 \]

\[ \pm \frac{1}{2} \pm \frac{3}{2} \pm \frac{5}{2} \pm \frac{9}{2} \pm \frac{15}{2} \pm \frac{45}{2} \]
\[(x-2) (x+4-i) (x+4+i) \]
\[= (x^2 + 8x + 17) \]
\[\frac{x^3 + 8x^2 + 17x}{x-2} \]
\[= x^3 + 6x^2 + x - 34 \]
Definition: A **rational function** is one that can be written in the form

$$f(x) = \frac{N(x)}{D(x)}$$

where $N(x)$ and $D(x)$ are polynomials and $D(x)$ is not the zero polynomial.

$$\begin{align*}
2x + 3 \\
-5x^0
\end{align*}$$

$$\begin{align*}
\frac{x^6 - 3x + 5}{5}
\end{align*}$$
\[ g(x) = \frac{1}{x^2 - 9} \]

\[ \text{Zoom Standard} \]
\[ x \to -3, \; y \to +\infty \]
\[ \frac{1}{(x-3)(x+3)} \]
\[ x = 3 \quad x = -3 \]
\[ \text{asymptotes} \]

\[ \text{Zoom Decimal} \]
\[ x \to 3^- \; y \to -\infty \]
\[ x \to 3^+ \; y \to +\infty \]
\[ x \to +\infty \; y \to 0 \]
\[ h(x) = \frac{x^2 - 4}{x + 2} \]

\[
\frac{(x-2)(x+2)}{(x+2)}
\]

\[ f(x) = x - 2 \quad x \neq -2 \]

\[ x \neq -2 \quad \text{discontinuous} \]

\[ \text{not asymptote} \]

\[ \text{hole} \]
**Asymptote**: a straight line approached by a given curve as one of the variables in the equation of the curve approaches infinity.
Definitions:
The line $x = a$ is a **vertical asymptote** of the graph of $f$ if $f(x) \to \infty$ or $f(x) \to -\infty$ as $x \to a$, either from the right or from the left.

The line $y = b$ is a **horizontal asymptote** of the graph of $f$ if $f(x) \to b$ as $x \to \infty$ or $x \to -\infty$. 
Asymptotes of a Rational Function

Let $f$ be the rational function

$$f(x) = \frac{N(x)}{D(x)} = \frac{a_n x^n + \ldots + a_1 x + a_0}{b_m x^m + \ldots + b_1 x + b_0}$$

where $N(x)$ and $D(x)$ have no common factors.

1. The graph of $f$ has **vertical** asymptotes at the zeros of $D(x)$.
2. The graph of $f$ has at most one horizontal asymptote determined by comparing the degrees of $N(x)$ and $D(x)$.
   a. If $n < m$, the graph of $f$ has the line $y = 0$ (the x-axis) as a horizontal asymptote.
   b. If $n = m$, the graph of $f$ has the line $y = \frac{a_n}{b_m}$ as a horizontal asymptote.
   c. If $n > m$, the graph of $f$ has no horizontal asymptote.
\[ f(x) = \frac{2x + 1}{x^2 + 1} \]

Vertical asymptote: \( x = -1 \)

Horizontal asymptote:

\[ f(x) = \frac{-4}{x^2 + 1} \]

Vertical asymptote: \( x = 1 \)

Horizontal asymptote: \( y = 0 \)

\[ f(x) = \frac{2}{(x - 1)^2} \]

Vertical asymptote: \( x = 1 \)

Horizontal asymptote: \( y = 0 \)

\[ \lim_{x \to -1} f(x) = \infty \]

\[ \lim_{x \to \infty} f(x) = 0 \]

\[ \lim_{x \to \infty} y = 2 \]

\[ y \to 2^+ \]

\[ x \to \infty, y \to 2^+ \]

\[ x \to -\infty, y \to 2^- \]
Guidelines for Graphing Rational Functions

Let \( f(x) = \frac{N(x)}{D(x)} \), where \( N(x) \) and \( D(x) \) are polynomials.

1. Simplify \( f \), if possible.
2. Find and plot the \( y \)-intercept (if any) by evaluating \( f(0) \).
3. Find the zeros of the numerator (if any) by solving the equation \( N(x) = 0 \). Then plot the corresponding \( x \)-intercepts.
4. Find the zeros of the denominator (if any) by solving the equation \( D(x) = 0 \). Then sketch the corresponding vertical asymptotes using dashed vertical lines.
5. Find and sketch the horizontal asymptote (if any) of the graph using a dashed horizontal line.
6. Plot at least one point \( \text{between} \) and one point \( \text{beyond} \) each \( x \)-intercept and vertical asymptote.
7. Use smooth curves to complete the graph between and beyond the vertical asymptotes.
\[ f(x) = \frac{x + 1}{x^2 - 1} \]

Horizontal asymptote is \( y = 0 \). The only vertical asymptote is \( x = 1 \). There will be a hole in the graph at \( x = -1 \).

\[ f(x) = \frac{x + 1}{x^2 - 1} = \frac{(x+1)}{(x+1)(x-1)} \]

\[ x \neq -1 \]

\[ \text{hole} \]

\[ \frac{1}{x-1} \quad x \neq 1 \quad VA: x = 1 \]

\[ x \to \infty \quad y \to 0 \quad HA: y = 0 \]

\[ (0, -1) \quad \text{y-intercept} \]
\[ g(x) = \frac{2x + 5}{4x - 6} \]

The horizontal asymptote is at \( y = \frac{1}{2} \), and the vertical asymptote is at \( x = \frac{3}{2} \).

\[ y\text{-intercept} \quad (0, \frac{5}{6}) \]
\[ x\text{-intercept} \quad \left(-\frac{5}{2}, 0\right) \]
\[ 2x + 5 = 0 \quad \Rightarrow \quad x = -\frac{5}{2} \]

\[ \frac{2x}{x} + \frac{5}{x} = \frac{4x}{x} - \frac{6}{x} \quad \Rightarrow \quad \frac{2 + \frac{5}{x}}{\frac{4 - \frac{6}{x}}{x}} = \frac{2}{\frac{4 - \frac{6}{x}}{x}} = \frac{1}{2} \]

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\[ h(x) = \frac{x^2}{x+1}. \]

No horizontal asymptote and a vertical asymptote at \( x = -1 \)

- \( y \)-intercept \((0, 0)\)
- Oblique \( x \)-intercept \((0, 0)\)
- \( VA: x = -1 \)
- \( HA: y = 0 \)
- \( HA: y = \frac{3}{5} \)
**Definition:** If the degree of the numerator is exactly one more than the degree of the denominator, the graph of the function has a **slant** or **oblique asymptote**. To find the equation of the slant asymptote, divide the denominator into the numerator.

If \( n = m + 1 \), then the graph of \( f \) has a slant asymptote at \( y = q(x) \), where \( q(x) \) is the quotient from the division algorithm.
Example Sketch the graph of \( y = \frac{x^2}{x - 2} = x + 2 + \frac{4}{x - 2} \)

- **y-Intercept:** (0, 0)
- **x-Intercept:** (0, 0)
- **Vertical asymptote:** \( x = 2 \)
- **Slant asymptote:** \( y = x + 2 \)

*Additional points:* \((-1/2, -0.1), (1, -1), (3, 9)\)
Rational Functions

1. Simplify
2. y-intercept
3. x-intercept let numerator = 0
4. VA: let denominator = 0 x = ?
5. HA: degree of N < deg of D \( y = 0 \)
   degree of N = deg of D \( y = \frac{a_n}{b_n} \)
6. SA: degree of N > deg of D divide \( y = \) quotient
4. \[ f(x) = \frac{x^2 + 5x + 8}{x + 3} \]
Find the equation of the quadratic that goes through the following points
Find the equation of the quadratic that goes through the following points:
A rectangular box with a square base is to be formed from a square piece of metal with 12-inch sides. A square piece with side $x$ is cut from the corners of the metal and the sides are folded up to form an open box. What value of $x$ will maximize the volume of the box?
The cost function is $C = 0.8x^2 - 17x + 300$. The revenue function is $R = -0.2x^2 + 39x$. Find when the profit is a maximum.
Example 1. Decide whether each data set would best be modeled by a linear model or a quadratic model.

a) (1, 3), (2, 5), (4, 6), (6, 8), (8, 9), (10, 10), (12, 13)

b) (2, 1), (4, 2), (6, 4), (8, 7), (9, 10), (11, 15), (13, 20)

Enter both sets of data into a graphing utility and display the scatter plots.
Example 2. The following table gives the mileage $y$, in miles per gallon, of a certain car at various speeds $x$ (in miles per hour).

a) Use a graphing utility to create a scatter plot of the data.
b) Use the regression feature of a graphing utility to find a quadratic model that best fits the data.
c) Use the model to predict the speed that gives the greatest mileage.

<table>
<thead>
<tr>
<th>Speed, $x$</th>
<th>Mileage, $y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>21.3</td>
</tr>
<tr>
<td>15</td>
<td>23.7</td>
</tr>
<tr>
<td>20</td>
<td>25.9</td>
</tr>
<tr>
<td>25</td>
<td>27.6</td>
</tr>
<tr>
<td>30</td>
<td>29.4</td>
</tr>
<tr>
<td>35</td>
<td>31.0</td>
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<tr>
<td>40</td>
<td>31.7</td>
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<tr>
<td>45</td>
<td>31.9</td>
</tr>
<tr>
<td>50</td>
<td>29.5</td>
</tr>
<tr>
<td>55</td>
<td>27.6</td>
</tr>
<tr>
<td>60</td>
<td>25.3</td>
</tr>
<tr>
<td>65</td>
<td>23.0</td>
</tr>
<tr>
<td>70</td>
<td>20.0</td>
</tr>
</tbody>
</table>
Example 3. For the data points below, determine whether a linear model or a quadratic model best fits the data.

(1, 5)
(2, 6)
(3, 8)
(4, 9)
(5, 11)
(6, 10)
(7, 11)
(8, 12)
(9, 14)
(10, 16)
The cost of producing $x$ units is $C = 0.25x^2 + 5x + 78$. The average cost per unit is

$$\bar{C} = \frac{0.25x^2 + 5x + 78}{x} = 0.25x + 5 + \frac{78}{x}.$$ 

Find the number of units that should be produced to minimize the average cost. Graph this function on a graphing utility, then use the “minimum” command. $\bar{C} \approx 17.66$
A game commission has determined that if 500 deer are introduced into a preserve, the population at any time \( t \) (in months) is given by

\[
N = \frac{500 + 350t}{1 + 0.2t}.
\]

What is the carrying capacity of the preserve?

The carrying capacity will be the horizontal asymptote, \( y = 1750 \).
Examples: (a) Find the domain of the function, (b) identify any horizontal and vertical asymptotes, and (c) verify your answer to part (a) by using a graphing utility and by creating a table of values.

1. \[ f(x) = \frac{1 - 5x}{1 + 2x} \]
2. \[ f(x) = \frac{3x^2 + x - 5}{x^2 + 1} \]
#36. (Page 294)  In a pilot project, a rural township is given recycling bins for separating and storing recyclable products. The cost \( C \) (in dollars) for supplying bins to \( p\% \) of the population is given by

\[
C = \frac{25,000p}{100 - p}, \quad 0 \leq p < 100.
\]

(a) Find the cost of supplying bins to 15\% of the population.

(b) Find the cost of supplying bins to 50\% of the population.

(c) Find the cost of supplying bins to 90\% of the population.
(d) Use a graphing utility to graph the cost function. Be sure to choose an appropriate viewing window. Explain why you chose the values that you used in your viewing window.

(e) According to this model, would it be possible to supply bins to 100% of the residents? Explain.
Example: \( f(x) = \frac{x + 1}{x} \).

**y-Intercept:** None

**x-Intercept:** (-1, 0)

**Vertical asymptote:** \( x = 0 \)

**Horizontal asymptote:** \( y = 1 \)

**Additional points:** (-2, 0.5), (-1.5, 1/3), (1, 2)
\[ g(x) = \frac{x - 2}{x^2 - 2x - 8} \]

**y-Intercept:** (0, -0.25)

**x-Intercept:** (2, 0)

*Vertical asymptote:* \( x = -2 \) and \( x = 4 \)

*Horizontal asymptote:* \( y = 0 \)

*Additional points:* (-4, -0.375), (0, 1/4), (6, 1/4)
\[ h(x) = \frac{x}{x^2 + 1} \]

**y-Intercept:** (0, 0)

**x-Intercept:** (0, 0)

**Vertical asymptote:** none

**Horizontal asymptote:** \( y = 0 \)

**Additional points:** \((-2, -0.4), (-1, -1/2), (1, 1/2)\)
Examples: Sketch the graph of the rational function by hand. Use a graphing utility to verify your graph.

1. \[ g(x) = \frac{x}{x^2 - 9} \]

<table>
<thead>
<tr>
<th>y-intercept</th>
<th>[ g(0) = 0 ]</th>
<th>Point: ((0, 0))</th>
</tr>
</thead>
<tbody>
<tr>
<td>x-intercepts</td>
<td>set ( x = 0 )</td>
<td>Point: ((0, 0))</td>
</tr>
<tr>
<td>vert. asymptotes</td>
<td>set ( x^2 - 9 = 0 )</td>
<td>( x = \pm 3 )</td>
</tr>
<tr>
<td>horiz. Asymptote</td>
<td>let ( x \to \pm\infty )</td>
<td>( y = 0 )</td>
</tr>
</tbody>
</table>
2. \[ f(x) = \frac{x + 4}{x^2 + x - 6} \]
3. \[ g(x) = \frac{3x^4 - 5x + 3}{x^4 + 1} \]