Method of Frobenius: The method of Frobenius consists of identifying a regular singular point \( x_0 \), substituting \( y = \sum c_n (x - x_0)^{n+r} \) into the given DE, and determining the unknown exponent \( r \) and the coefficients \( c_n \).

In all examples, we will assume that \( x_0 = 0 \).
After substituting $y = \sum c_n x^{n+r}$ into the DE and simplifying, the indicial equation is a quadratic equation in $r$ that results from equating the total coefficient of the lowest power of $x$ to zero. (Don’t memorize formulas.)
Cases of Indicial Roots:
Assume $r_1, r_2$ are real solutions to the indicial equation and $r_1$ is the larger root.

Case 1: Roots not differing by an integer
There will be two linearly independent solutions of the DE.
Case 2: **Roots differing by a positive integer**

There will be two linearly independent solutions of the form

\[ y_1 = \sum c_n x^{n+\eta_1}, c_0 \neq 0 \]

\[ y_2 = Cy_1(x) \ln x + \sum_{n=0}^{\infty} b_n x^{n+\eta_2}, b_0 \neq 0 \ (C \text{ may be } 0.) \]
Case 3:  **Equal indicial roots**

There will be two linearly independent solutions of the form

\[ y_1 = \sum c_n x^{n+\eta}, \ c_0 \neq 0 \]

\[ y_2 = y_1(x) \ln x + \sum_{n=1}^{\infty} b_n x^{n+\eta}, \ b_0 \neq 0 \]
Example 4: Find 2 linearly independent solutions to the given differential equation.

\[ xy'' + y = 0 \]

\[ x = 0 \text{ is a singular point} \]

Note: \[ y'' + \frac{1}{x} y = 0 \]
\[ Q(x) = \frac{1}{x} \Rightarrow x = 0 \]

is a regular singular pt.

Assume: \[ y = \sum_{n=0}^{\infty} c_n x^{n+r} \]
\[ y' = \sum_{n=0}^{\infty} c_n (n+r)x^{n+r-1} \]
\[ y'' = \sum_{n=0}^{\infty} c_n (n+r)(n+r-1)x^{n+r-2} \]
\[
xy'' + y' = 0
\]

\[
x \sum_{n=0}^{\infty} c_n (n+r-1)(n+r)x^{n+r-2} + \sum_{n=0}^{\infty} c_n x^n = 0
\]

Let \( k = n \)

\[
\sum_{k=0}^{\infty} c_k (k+r-1)(k+r)x^{k+r-1} + \sum_{k=1}^{\infty} c_k x^k = 0
\]

\( k = 0 \)

\[
c_0 (r-1)(r)x^{r-1} = 0 \quad x^{r-1} \text{ is not identically 0} \quad c_0 \text{ is the first nonzero coefficient}
\]

\( \Rightarrow (r-1)(r) = 0 \) (Indicial Equation)

\( r = 1, 0 \) (indicial roots)
\[ \sum_{k=0}^{\infty} c_k (k+r-1)(k+r) X^k + \sum_{k=1}^{\infty} c_k X^k = 0 \]

Let \( r = 1 \)

\[ \sum_{k=1}^{\infty} [c_k (k+1)(k+2) + c_{k-1}] X^k = 0 \]

\[ \forall k \geq 1, \quad c_k (k+1)(k+2) + c_{k-1} = 0 \]

\[ \iff \quad c_k = \frac{-c_{k-1}}{k(k+1)} \]

\[
\begin{align*}
k = 1 & \quad c_1 = \frac{-c_0}{1(2)} = \frac{-c_0}{2} \\
k = 2 & \quad c_2 = \frac{-c_1}{2(3)} = \frac{-(-c_0/2)}{2 \cdot 3} = \frac{c_0}{12} \\
k = 3 & \quad c_3 = \frac{-c_2}{3(4)} = \frac{-c_0/12}{3 \cdot 4} = -\frac{c_0}{144}
\end{align*}
\]

The Frobenius solution corresponding to \( r = 1 \) is

\[ y = c_0 X - \frac{c_0}{2} X^2 + \frac{c_0}{12} X^3 - \frac{c_0}{144} X^4 + \ldots \]

\[ = c_0 X \left[ 1 - \frac{X}{2} + \frac{X^2}{12} - \frac{X^3}{144} + \ldots \right] \]
\[ \sum_{k=0}^{\infty} c_k (k+r-1)(k+r) x^{k+r-1} + \sum_{k=1}^{\infty} c_k x^k = 0 \]

\[ c_0 (r-1) x^{r-1} + \sum_{k=1}^{\infty} \left[ c_k (k+r)(k+r)+c_{k-1} \right] x^k = 0 \]

\[ r=0 \]

\[ \sum_{k=1}^{\infty} \left[ c_k (k+r-1) + c_{k-1} \right] x^{k-1} = 0 \]

\[ \forall k \geq 1, \ c_k (k+r-1) + c_{k-1} = 0 \]

\[ \iff \]

\[ c_k = -\frac{c_{k-1}}{(k-1)r} \]

\[ k=1 \]

\[ c_1 = \frac{C_0}{0 \cdot 1} \quad \text{no recurrence formula} \]

\[ k=2 \]

\[ c_2 = -\frac{c_1}{1 \cdot 2} \]

\[ k=3 \]

\[ c_3 = -\frac{c_2}{2 \cdot 3} = -\frac{-c_1}{2 \cdot 3} = \frac{c_1}{12} \]

\[ k=4 \]

\[ c_4 = -\frac{c_3}{3 \cdot 4} = -\frac{-c_1}{3 \cdot 4} = \frac{c_1}{144} \]

Frobenius type solution corresponding to \( r=0 \)

\[ y = c_1 x - \frac{c_1}{12} x^2 + \frac{c_1}{144} x^4 + \cdots \]

\[ = c_1 x \left[ 1 - \frac{x^2}{2} + \frac{x^4}{12} - \frac{x^6}{144} + \cdots \right] \]

The 2 solutions are linearly dependent.
Case 2: Roots differing by a positive integer

There will be two linearly independent solutions of the form

\[ y_1 = \sum c_n x^{n+1}, c_0 \neq 0 \]

\[ y_2 = Cy_1(x) \ln x + \sum_{n=0}^{\infty} b_n x^{n+2}, b_0 \neq 0 \quad (C \text{ may be } 0.) \]

Here, \( y_1 = c_0 x \left[ 1 - \frac{x}{2} + \frac{x^2}{12} - \frac{x^3}{144} + \cdots \right] \)

Using variation of parameters where we assume 2nd solution has form

\[ y_2(x) = y_1(x) \int \frac{e^{\int P(x) dx}}{y_1^2(x)} dx \quad y'' + P(x)y + Q(x)y = 0 \]

\[ y_1(x) \int e^{\int P(x) dx} \frac{dy_1}{y_1^2(x)} dx \quad [\text{let } C_0 = 1] \]

\[ = y_1(x) \int \frac{1}{\left[ \frac{x - \frac{x^2}{2} + \frac{x^3}{12} - \frac{x^4}{144} + \cdots}{x} \right]^2} dx \]
\[ = y_1(x) \int \left( \frac{1}{X^2 - X^3 + \frac{5}{12}X^4 - \frac{X^5}{12} + \ldots} \right) \, dx \]

Find \( \frac{1}{1 - X + \frac{5}{12}X^2 - \ldots} \)

by long division.
\[ y_1(x) \left( \frac{\ dx}{x^2 \left( 1 - x + \frac{5}{6} x^2 - \frac{x^3}{12} + \ldots \right)} \right) \]
\[ = y_1(x) \left( \int \frac{1}{x^2} \left[ 1 + x + \frac{7}{12} x^2 + \ldots \right] \, dx \right) \]
\[ = y_1(x) \left[ \left( \frac{1}{x^2} + \frac{1}{x} + \frac{7}{12} x + \ldots \right) \right] \]
\[ = y_1(x) \left[ -\frac{1}{x} + \ln x + \frac{7}{12} x + \ldots \right] \]
\[ = \left[ x - \frac{x^3}{2} + \frac{x^4}{12} - \frac{x^5}{40} + \ldots \right] \left[ -\frac{1}{x} + \ln x + \frac{7}{12} x + \ldots \right] \]
\[ = y_1(x) \ln x + \left( x \ln x + \sum_{n=1}^{\infty} b_n x^n \right) \]
\[ y_2(x) = y_1(x) \ln x + \sum_{n=0}^{\infty} b_n x^n \]

(has the form \( \sum_{n=0}^{\infty} b_n x^n \))
MAT223 Exam Review

Determine if a DE is linear or nonlinear and how many arbitrary constants are in the general solution.

Sketch the phase portrait for a DE of the form \( \frac{dx}{dt} = x(a - bx) \).

Identify the trajectory of a periodic solution.

Verify a particular solution to a 2\(^{nd}\) order linear DE.

Find the general solution to a 2\(^{nd}\) order homogeneous linear DE.

Find the particular solution to a 2\(^{nd}\) order linear DE.
Given the graph of $f(t)$, find the graph of a function involving $f(t-a)$ and/or $U(t-a)$.

Eliminate one variable from a system of linear first-order DE’s.

Evaluate 3 Laplace transforms.

Solve 2 DE’s by separation of variables or a method of your choice.

Define local truncation error.

Use a Wronskian to determine whether or not a set of functions is linearly independent.
Sketch a trajectory for a 2nd order linear DE.
For this same DE, classify (0, 0) as unstable, stable (not asymptotically stable), or asymptotically stable.
Find the characteristic equation for a system of DE’s.
For the same system, find the eigenvalues and eigenvectors of the matrix of coefficients.
Given the DE and formulas for $W, W_1, W_2$, use the method of variation of parameters to find solution.
Set up the DE for an a spring problem.

Given $y_c$ for a linear DE, what functions should be tried for $y_p$ using the method of undetermined coefficients.

Describe the difference between Euler and Runge-Kutta.
Find the recurrence formula for the power series solution to a DE. (about an ordinary point)

Given the recurrence formula, find a few coefficients.

Find the singular points of a DE.

Determine whether a given singular point is regular or not.

Find the indicial roots for a Frobenius-type solution.