Examples: Find two linearly independent power series solutions about the ordinary point $x = 0$.

2. $y'' - (x + 1)y' - y = 0.$

$$y = \sum_{k=0}^{\infty} c_k x^k$$
$$y' = \sum_{k=0}^{\infty} kc_k x^{k-1}$$
$$y'' = \sum_{k=0}^{\infty} kc_k (k-1) x^{k-2}$$

$$\sum_{k=0}^{\infty} c_k x^k = \sum_{k=0}^{\infty} c_k x^k - \sum_{k=0}^{\infty} c_k x^k$$

$$\sum_{k=0}^{\infty} c_k x^k = \sum_{k=0}^{\infty} c_k x^k - \sum_{k=0}^{\infty} c_k x^k$$

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$$\sum_{k=0}^{\infty} c_k x^k = \sum_{k=0}^{\infty} c_k x^k - \sum_{k=0}^{\infty} c_k x^k$$

$$\sum_{k=0}^{\infty} c_k x^k = \sum_{k=0}^{\infty} c_k x^k - \sum_{k=0}^{\infty} c_k x^k$$
\[
\sum_{n=0}^{\infty} c_{n+2} (n+2)(n+1)X^n - \sum_{n=0}^{\infty} c_n X^n = 0
\]

\[
- \sum_{n=0}^{\infty} c_{n+1} (n+1)X^n - \sum_{n=0}^{\infty} c_n X^n = 0
\]

\[
\left[ c_2 (2)(1) - c_1 (1) - c_0 \right]
\]

\[
+ \sum_{n=1}^{\infty} \left[ c_{n+2} (n+2)(n+1) - c_n - c_{n+1} (n+1) - c_n \right] X^n = 0
\]

\[
\left[ 2c_2 - c_1 - c_0 \right]
\]

\[
+ \sum_{n=1}^{\infty} \left[ c_{n+2} (n+2)(n+1) - c_{n+1} (n+1) - c_n (n+1) \right] X^n = 0
\]

\[
\begin{align*}
0 & 
2c_2 - c_1 - c_0 = 0 \\
0 & 
2c_2 = c_1 + c_0 \\
c_2 & = \frac{c_1 + c_0}{2}
\end{align*}
\]

\[
\sum_{n=0}^{\infty} c_{n+2} (n+2)(n+1)X^n - \sum_{n=0}^{\infty} c_n X^n = 0
\]

\[
\Rightarrow (n+1) \left[ c_{n+2} (n+2)(n+1) - c_{n+1} (n+1) - c_n (n+1) \right] = 0
\]

\[
\Rightarrow (n+1) c_{n+2} (n+2) - c_{n+1} (n+1) - c_n (n+1) = 0
\]

\[
(\because (n+1)) \quad c_{n+2} (n+2) - c_{n+1} - c_n = 0
\]

\[
c_{n+2} (n+2) = c_{n+1} + c_n
\]

\[
c_{n+2} = \frac{c_{n+1} + c_n}{n+2}
\]

\text{recurrence formula}
Our arbitrary constants will be $C_0$ and $C_1$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$C_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$C_2 = \frac{C_1 + C_0}{2}$</td>
</tr>
<tr>
<td>1</td>
<td>$C_3 = \frac{C_2 + C_1}{3} = \frac{(\frac{C_1 + C_0}{2}) + C_1}{3} = \frac{3C_1 + C_0}{6}$</td>
</tr>
<tr>
<td>2</td>
<td>$C_4 = \frac{C_3 + C_2}{4} = \frac{\left(\frac{3C_1 + C_0}{6}\right) + \left(\frac{C_1 + C_0}{2}\right)}{4} = \frac{3C_1 + 2C_0}{12}$</td>
</tr>
<tr>
<td>3</td>
<td>$C_5 = \frac{C_4 + C_3}{5} = \frac{\left(\frac{3C_1 + 2C_0}{12}\right) + \left(\frac{3C_1 + C_0}{6}\right)}{5} = \frac{9C_1 + 4C_0}{60}$</td>
</tr>
</tbody>
</table>
\[
y = c_0 + c_1 x + \left( \frac{c_1 + c_0}{2} \right) x^2 + \left( \frac{3c_1 + c_0}{6} \right) x^3 + \left( \frac{3c_1 + 2c_0}{12} \right) x^4 + \left( \frac{9c_1 + 4c_0}{60} \right) x^5 + \cdots
\]

\[
y = c_0 \left[ 1 + \frac{1}{2} x + \frac{1}{6} x^3 + \frac{1}{6} x^4 + \frac{1}{15} x^5 + \cdots \right] + c_1 \left[ x + \frac{1}{2} x^2 + \frac{1}{2} x^3 + \frac{1}{4} x^4 + \frac{3}{20} x^5 + \cdots \right]
\]

\[y'' - (x+1)y' - y = 0 \quad \text{(no singular points)}\]

\[y'' + P(x)y' + Q(x)y = 0\]

R = distance to nearest singular point

Here, R = \infty

The series solution converges for \((-\infty, \infty)\).
Example: Section 6.1 #12

\[(x^2-1)y'' + 4xy' + 2y = 0\]
\[\frac{y''}{(x^2-1)} + \frac{4x}{(x^2-1)}y' + \frac{2}{(x^2-1)}y = 0\]

There are singularities when
\[x^2-1 = 0 \iff x = -1, 1\]
\[\Rightarrow R = 1\]
A power series solution centered at 0 will converge at least on \((-1, 1)\).

Solution: \(y_1 = 1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \ldots\)
\[y_2 = x\]
6.2 Solutions about Singular Points

It is not always possible to find a solution of the form \( \sum c_n (x - x_0)^n \) about a singular point \( x_0 \); it may be possible to find a solution of the form \( \sum c_n (x - x_0)^{n+r} \) where \( r \) is a constant. If \( r \) is not a nonnegative integer, \( \sum c_n (x - x_0)^{n+r} \) is not a power series.
Definitions.
A singular point $x_0$ is said to be a regular singular point of the differential equation $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$ if both $(x - x_0)P(x)$ and $(x - x_0)^2 Q(x)$ are analytic at $x_0$. A singular point that is not regular is said to be an irregular singular point of the equation.

[If $(x - x_0)$ appears at most once in the denominator of $P(x)$ and at most twice in the denominator of $Q(x)$, then $x = x_0$ is a regular singular point.]
Examples: Determine the singular points and classify them as regular or irregular.

\[ \rho(x) = \frac{1}{x} \quad \text{singular points} \quad x = 0 \]

1. \[ y'' - \frac{1}{x} y' + \frac{1}{(x-1)^3} y = 0 \]
\[ Q(x) = \frac{1}{(x-1)^3} \quad x = 1 \]

0 is a regular singular point
1 is an irregular singular point

\[ x \rho(x) = x \left( -\frac{1}{x} \right) = -1 \quad \text{(trivially analytic at 0)} \]

\[ (x-1)^2 Q(x) = (x-1)^2 \left( \frac{1}{(x-1)^3} \right) = \frac{1}{x-1} \]

which is not analytic at 1
\[ x(x^2+1)^2 y'' + y = 0 \quad a_0(x)y'' + a_1(x)y' + a_2(x)y = 0 \]

2.
\[ y'' + \frac{1}{x(x^2 + 1)^2} y = 0 \quad y'' + p(x)y' + q(x)y = 0 \]
\[ a_1(x) = P(x) = 0 \]

\[ 0, \pm i \text{ are regular singular points} \]

\[ Q(x) = \frac{1}{x(x^2+1)^2} \]
\[ X = 0 \]
\[ x^2 + 1 = 0 \]
\[ x = \pm i \]

\[ X = 0 \pm i \]

If you found a power series solutions to this DE centered at 1, what is \( R \)?

\( 0 \) is closest singular point.
Theorem 6.2  Frobenius’ Theorem
If \( x = x_0 \) is a regular singular point of the differential equation
\[ a_2(x)y'' + a_1(x)y' + a_0(x)y = 0, \]
then there exists at least one solution of the form
\[
y = (x - x_0)^r \sum_{n=0}^{\infty} c_n (x - x_0)^n = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r}
\]
where the number \( r \) is a constant to be determined. The series will converge at least on some interval \( 0 < x - x_0 < R \).
Method of Frobenius: The method of Frobenius consists of identifying a regular singular point $x_0$, substituting $y = \sum c_n (x - x_0)^{n+r}$ into the given DE, and determining the unknown exponent $r$ and the coefficients $c_n$.

In all examples, we will assume that $x_0 = 0$. 
Example 3: Solve by the method of Frobenius.

\[ 2x^2 y'' - xy' + (x^2 + 1)y = 0 \]

\[ y'' - \frac{1}{2x} y' + \frac{(x^2 + 1)}{2x^2} y = 0 \]

\[ P(x) = \frac{1}{2x}, \quad Q(x) = \frac{x^2 + 1}{2x^2} \]

\[ y = \sum c_n x^{n+r} \]

Assume:

\[ y' = \sum (n+r)c_n x^{n+r-1} \]

\[ y'' = \sum (n+r-1)(n+r)c_n x^{n+r-2} \]

\[ \Rightarrow 0 \text{ is a regular singular point} \]

Assume that \( c_0 \) is the first non-zero coefficient

(any coefficients such as \( c_2, c_{-1} \), are therefore 0)
\[2x^2 y'' - xy' + xy + \sum_{n=0}^{\infty} c_n x^{n+r} = 0\]

\[2\sum_{n=0}^{\infty} (n+r-1)(n+r)c_n x^{n+r} - \sum_{k=0}^{n} (n+r)c_k x^{n+r} - \sum_{k=0}^{n} c_k x^{n+r+2} = 0\]

\[2\sum_{k=2}^{\infty} (k+r-1)(k+r)c_k x^{k+r} - \sum_{k=0}^{\infty} (k+r)c_k x^{k+r} - \sum_{k=0}^{\infty} c_k x^{k+r+2} = 0\]
\[ \sum_{k=0}^{r-1} \frac{(k+r-1)!}{(k+r)!} c_k x^{r-k} = \sum_{k=0}^{r} \frac{r!}{k!} c_k x^{k} = 0 \]

\[ \sum_{k=2}^{r} \frac{c_k}{k} x^{k} + \sum_{k=0}^{r} c_k x^{k} = 0 \]

\[ k=0 \]

\[ (2(r-1)(r)c_0 - rc_0 + c_0) x^r = 0 \quad c_0 \neq 0 \]

\[ \Rightarrow 2(r-1)(r)c_0 - rc_0 + c_0 = 0 \quad \left( \div c_0 \right) \]

\[ 2(r-1)(r) - r + 1 = 0 \quad \text{(indicial equation)} \]

\[ 2r^2 - 2r - r + 1 = 0 \]

\[ 2r^2 - 3r + 1 = 0 \]

\[ (2r-1)(r-1) = 0 \]

\[ r = \frac{1}{2}, 1 \]

indicial roots
\[ 2 \sum_{k=0}^{\infty} (k+1) k^r c_k x^k - \sum_{k=0}^{\infty} (k+1) c_k x^k + \sum_{k=2}^{\infty} c_k x^k = 0 \]

Let \( r = 1 \)

\[ 2 \sum_{k=0}^{\infty} k(k+1) c_k x^k - \sum_{k=0}^{\infty} (k+1) c_k x^k \]

\[ + \sum_{k=2}^{\infty} c_k x^k = 0 \]

\( k = 0 \)

\[ 0 - c_0 x + c_0 x = (-c_0 + c_0) x = 0 \]

\( k = 1 \)

\[ 2 \times 1(2) c_1 x^2 - 2 c_1 x^2 + c_1 x^2 = 0 \]

\[ \Rightarrow 3 c_1 x^2 = 0 \quad c_1 = 0 \]

\( k \geq 2 \)

\[ 2k(k+1)c_k - (k+1)c_k + c_{k-2} + c_k \]

\[ \Rightarrow 2k(k+1)c_k - (k+1)c_k + c_{k-2} + c_k = 0 \]

\( c_0 \) can be any number
\[ 2k^2 c_k + 2k c_k - k c_k - c_k + c_{k-2} + c_k = 0 \]

\[ 2k^2 + k \]

\[ c_k = -c_{k-2} \]

recurrence formula

\[ k \geq 2 \]

\[ c_k = -\frac{c_{k-2}}{k(2k+1)} \]

\[ k = 2 \]

\[ c_2 = -\frac{c_0}{2(5)} = -\frac{c_0}{10} \]

\[ k = 3 \]

\[ c_3 = -\frac{c_1}{3(5)} = 0 \]

\[ k = 4 \]

\[ c_4 = -\frac{c_2}{4(9)} = -\left(\frac{-c_0}{10}\right) = \frac{c_0}{36} \]

\[ \text{etc.} \]

Corresponding to \( r = 1 \), we get the solution:

\[ y = c_0 x - \frac{c_0}{10} x^3 + \frac{c_0}{360} x^5 - \ldots \]

\[ = c_0 x \left[ 1 - \frac{1}{10} x^2 + \frac{1}{360} x^4 - \ldots \right] \]
For $r = \frac{1}{2}$ ...

$C_k = \frac{-C_{k-2}}{k(2k-1)}$

$k = 2 \quad C_2 = \frac{-C_0}{2(3)} = -\frac{C_0}{6}$

$k = 3 \quad C_3 = \frac{-C_0}{3(5)} = 0$

$k = 4 \quad C_4 = \frac{-C_2}{4(7)} = -\frac{(-\frac{C_0}{6})}{28} = \frac{C_0}{168}$

et cetera.

$c_0$ for $r = \frac{1}{2} \quad c_0$ for $r = 1$

$y = C_1x^2[1 - \frac{x^2}{6} + \frac{x^4}{168} - \frac{x^6}{11088} + ...] + C_2x[1 - \frac{x^2}{10} + \frac{x^4}{360} - ...]$