Section 4.1 (Homework)

#19. \( X_1 = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} ; X_2 = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} \)

\[ X_3 = \begin{pmatrix} 3 \\ -6 \\ 12 \end{pmatrix} + t \begin{pmatrix} 2 \\ 4 \\ 4 \end{pmatrix} \]

By inspection: \( X_3 = 2X_1 + X_2 \)

\[ \{X_1, X_2, X_3\} \text{ is not linearly independent and therefore not a fundamental set} \]
OR: 

\[ W(x_1, x_2, x_3) = \begin{vmatrix} 1+t & 1 & 3+2t \\ -2+2t & -2 & -6+4t \\ 4+2t & 4 & 12+4t \end{vmatrix} \]

\[ = (1+t)(-2)(12+4t) + 1(-6+4t)(4+2t) + (3+2t)(-2+2t)(4) - (4+2t)(2)(3+2t) - 4(-6+4t)(1+t) - (12+4t)(-2+2t)(1) \]

\[ = -24 - 8t - 24t - 8t^2 - 24 - 12t + 16t + 8t^2 - 24 + 24t - 16t + 16t^2 + 12t + 8t^2 + 24 + 24t - 16t - 16t^2 + 24 - 24t + 8t - 8t^2 = 0 \]

\[ \implies \{ x_1, x_2, x_3 \} \text{ is linearly dependent} \]

\[ \implies \text{not a fundamental set} \]
Case 2: Only one eigenvector corresponding to a double eigenvalue.

Example 3: Find the general solution of the given system.

\[
\frac{dx}{dt} = 12x - 9y \\
\frac{dy}{dt} = 4x
\]

\[
A = \begin{pmatrix} 12 & -9 \\ 4 & 0 \end{pmatrix}
\]

\[
\lambda = 6; \ K = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \\
X_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{6t}
\]
\[ X_2 = Ke^{\lambda t} + Pe^{\lambda t} \]
\[ X'_2 = Kt\lambda e^{\lambda t} + Ke^{\lambda t} + P\lambda e^{\lambda t} \]
\[ X'_2 = AX_2 \]

**Solve for \( P \):**

\[ Kt\lambda e^{\lambda t} + Ke^{\lambda t} + P\lambda e^{\lambda t} = AKte^{\lambda t} + APe^{\lambda t} \]

\[ (AK - \lambda K)te^{\lambda t} + (AP - \lambda P - K)e^{\lambda t} = 0 \]

\[ (A - \lambda I)K = 0, (A - \lambda I)P = K \]

\[ (A - \lambda I)P = K \]

\[
\begin{pmatrix}
6 & -9 \\
4 & -6 \\
\end{pmatrix}
\begin{pmatrix}
p_1 \\
p_2 \\
\end{pmatrix} =
\begin{pmatrix}
3 \\
2 \\
\end{pmatrix} \Rightarrow \ldots \Rightarrow P = 
\begin{pmatrix}
2 \\
1 \\
\end{pmatrix}
\]
\begin{align*}
X_2 &= Kte^{\lambda_1 t} + Pe^{\lambda_1 t} \\
X_2 &= \binom{3}{2}te^{6t} + \binom{2}{1}e^{6t} \\
X &= c_1 \binom{3}{2}e^{6t} + c_2 \left[ \binom{3}{2}te^{6t} + \binom{2}{1}e^{6t} \right]
\end{align*}

(\text{general solution})

\textcolor{red}{X_1} \quad \textcolor{red}{X_2}
Complex Eigenvalues

Theorem 4.8  Solutions Corresponding to a Complex Eigenvalue

Let $A$ be the coefficient matrix having real entries of the homogeneous system and let $K_1$ be an eigenvector corresponding to the complex eigenvalue $\lambda_1 = \alpha + \beta i$, $\alpha, \beta$ real. Then $K_1 e^{\lambda_1 t}, \bar{K}_1 e^{\lambda_1^* t}$ are solutions of the homogeneous system. $\bar{K}_1$ has entries that are conjugates of the entries of $K_1$. 
Theorem 4.9  Real Solutions Corresponding to a Complex Eigenvalue

Let $\lambda = \alpha + \beta i$ be a complex eigenvalue of the coefficient matrix $A$ in the homogeneous system and let 

$$B_1 = \frac{1}{2}(K_1 + \overline{K_1}); \quad B_2 = \frac{i}{2}(-K_1 + \overline{K_1})$$

be column vectors.

Then

$$X_1 = [B_1 \cos \beta t - B_2 \sin \beta t]e^{\alpha t}$$

$$X_2 = [B_2 \cos \beta t + B_1 \sin \beta t]e^{\alpha t}$$

are linearly independent solutions on $(-\infty, \infty)$.

**Note:** In the exercises involving complex eigenvalues, the form of the answer will vary according to the eigenvector chosen.
Example 5: Find the general solution of the given system.

\[
\begin{align*}
\frac{dx}{dt} &= 4x + 5y \\
\frac{dy}{dt} &= -2x + 6y
\end{align*}
\]

\[A = \begin{pmatrix} 4 & 5 \\ -2 & 6 \end{pmatrix}\]

\[\lambda = 5 \pm 3i \quad \text{Using } \lambda = 5 \pm 3i, \text{ we obtain } \begin{pmatrix} -5 \\ -1 - 3i \end{pmatrix} \]

\[
\begin{align*}
(-1 - 3i)k_1 + 5k_2 &= 0 \\
-2k_1 + (1 - 3i)k_2 &= 0
\end{align*}
\]

... \[\begin{pmatrix} -1 - 3i \\ 5 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \]

\[\begin{align*}
-3k_1 + 5k_2 &= 0 \\
-2k_1 + k_2 - 3k_2 &= 0
\end{align*}\]

\[\begin{pmatrix} -1 - 3i \\ 5 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{(multiply by 2)}
\]

\[\begin{pmatrix} 2 + 6i \\ -2 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{(multiply by } -3i + 3i + 9i) \]

\[\begin{pmatrix} 2 + 6i \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} + \begin{pmatrix} -10k_2 \end{pmatrix} = \begin{pmatrix} 0 \end{pmatrix} \quad \text{same equation}
\]

\[k_2 = \frac{1}{10} (2 + 6i)k_1 = \frac{1 + 3i}{5} k_1\]
Let \[ k_2 = \frac{(1+3i)}{5} k_1 \text{ or } \left(\frac{5}{1+3i}\right) k_2 = k_1, \]
\[ k_2 = -1 - 3i \implies k_1 = -5 \]

\[ K_1 = \begin{pmatrix} -5 \\ -1 - 3i \end{pmatrix}; \bar{K}_1 = \begin{pmatrix} -5 \\ -1 + 3i \end{pmatrix} \]

\[ B_1 = \frac{1}{2} (K_1 + \bar{K}_1) = \frac{1}{2} \begin{pmatrix} -10 \\ -2 \end{pmatrix} = \begin{pmatrix} -5 \\ -1 \end{pmatrix} \]

\[ B_2 = \frac{i}{2} (-K_1 + \bar{K}_1) = \frac{i}{2} \begin{pmatrix} 0 \\ 6i \end{pmatrix} = \begin{pmatrix} 0 \\ 3i^2 \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \end{pmatrix} \]

\[ X_1 = \left[ \begin{pmatrix} -5 \\ -1 \end{pmatrix} \cos 3t - \begin{pmatrix} 0 \\ 3 \end{pmatrix} \sin 3t \right] e^{5t} \]

\[ X_2 = \left[ \begin{pmatrix} 0 \\ -3 \end{pmatrix} \cos 3t + \begin{pmatrix} -5 \\ -1 \end{pmatrix} \sin 3t \right] e^{5t} \]
4.5.1 Phase Portraits and Stability for Linear Systems

For autonomous systems of first-order equations of the form

\[
\frac{dx}{dt} = f(x, y) \\
\frac{dy}{dt} = g(x, y)
\]

1. A critical point is a point \((x_1, y_1)\) for which

\[ f(x_1, y_1) = g(x_1, y_1) = 0. \]

We assume that the critical point is isolated and that the functions \(f\) and \(g\) are continuous and have continuous first partial derivatives in a neighborhood of the critical point.

2. Such a system is called a plane autonomous system since a solution of the system can be interpreted as a parametrized curve or trajectory in the \(xy\)-plane or phase plane.
The systems of equations that we solved in order to solve an autonomous linear second order equation in Chapter 3 \((ax'' + bx' + cx = 0)\) are special cases of the general autonomous linear system of two first-order equations

\[
\begin{align*}
\frac{dx}{dt} &= ax + by \\
\frac{dy}{dt} &= cx + dy
\end{align*}
\]

Let \(y = x'\)

\[
\begin{align*}
\frac{dy}{dt} &= x'' = \frac{-b}{a}x' - \frac{c}{a}x \\
\frac{dx}{dt} &= y; \frac{dy}{dt} = -\frac{c}{a}x - \frac{b}{a}y
\end{align*}
\]

Just as in Chapter 3, there are 4 types of critical points: node, saddle point, spiral point, or center and 3 classifications of stability: asymptotically stable, stable (not asymptotically stable), or unstable. The type of critical point and the type of stability are determined by characteristics of the eigenvalues of the coefficient matrix. (See pages 279 – 285).
Example:

Classify the critical point (0, 0) and obtain the phase portrait of the system:

\[
\begin{align*}
\frac{dx}{dt} &= -x + 2y \\ 
\frac{dy}{dt} &= -7x + 8y
\end{align*}
\]

This is Case 1. There are two distinct real positive eigenvalues. According to Theorem 4.10 iii.) on page 285, the critical point (0, 0) is unstable (an unstable node).
t0 = 0.
y1' = -y1 + y2
y1 =
y2' = -7 \cdot y1 + 8 \cdot y2
y2 =
y3' =
y3 =
y4 =

CHECK AXES FOR 2ND-ORDER SYSTEM
Use F8.
restart;

\[ DE1 := \frac{d}{dt} x(t) = -x(t) + 2y(t); \]
\[ DE2 := \frac{d}{dt} y(t) = -7x(t) + 8y(t); \]

\[ DE1 := \frac{d}{dt} x(t) = -x(t) + 2y(t) \quad (1) \]
\[ DE2 := \frac{d}{dt} y(t) = -7x(t) + 8y(t) \quad (2) \]

\[ \text{dolve[interactive]}((DE1, DE2)); \]
\[\frac{d}{dt} x(t) = -x(t) + 2y(t)\]
\[\frac{d}{dt} y(t) = -7x(t) + 8y(t)\]

\[x(0) = 1\]
\[y(0) = 1\]

\{y(t) = e^t, x(t) = e^t\}
\( y(t) = e^t, \quad x(t) = e^t \)
Case II: asymptotically stable node
5.1 The Laplace Transform

\[
\frac{d}{dx} \int_{a}^{b} dx, \int_{a}^{b} dx \text{ can be considered to be transforms because they transform one function into another. All 3 of these possess the linearity properties. That is,}
\]

\[
\frac{d}{dt}[\alpha f(t) + \beta g(t)] = \alpha \frac{d[f(t)]}{dt} + \beta \frac{d[g(t)]}{dt}
\]

\[
\int[\alpha f(t) + \beta g(t)] dt = \alpha \int f(t) dt + \beta \int g(t) dt
\]

\[
\int_{0}^{3} 3t^2 y dt = 27 y = F(y)
\]

Note:

\[
\int_{0}^{b} K(s,t) f(t) dt = \lim_{b \to \infty} \int_{0}^{b} K(s,t) f(t) dt = F(s)
\]
Definition 5.1  Laplace Transform
Let \( f \) be a function defined for \( t \geq 0 \). The integral

\[
L\{f(t)\} = \int_{0}^{\infty} e^{-st} f(t) \, dt
\]

is said to be the Laplace transform of \( f \) provided the integral converges.

We write

\[
L\{f(t)\} = F(s); \quad L\{g(t)\} = G(s); \quad L\{y(t)\} = Y(s), \text{ etc.}
\]
Example 1:

\[
L\{1\} = \int_0^\infty e^{-st} (1) \, dt = \lim_{b \to \infty} \int_0^b e^{-st} \, dt
\]

\[
\left. \frac{du}{s} = -s \, dt \right|_{u = -st}
\]

\[
\lim_{b \to \infty} \left[ -\frac{e^{-st}}{s} \right]_0^b = \lim_{b \to \infty} \frac{-e^{-sb} + 1}{s} = \frac{1}{s}, \quad s > 0
\]

\[
\mathcal{L}\{1\} = \frac{1}{s}, \quad s > 0
\]
The Laplace transform is a **linear** transform because it is defined as an integral.

\[
\int_0^\infty e^{-st} [\alpha f(t) + \beta g(t)] \, dt = \alpha \int_0^\infty e^{-st} f(t) \, dt + \beta \int_0^\infty g(t) \, dt
\]

\[
\mathcal{L}\{\alpha f(t) + \beta g(t)\} = \alpha \mathcal{L}\{f(t)\} + \beta \mathcal{L}\{g(t)\}
\]

whenever both integrals on the right converge for \(s > 0\).
Theorem 5.1  Transforms of Some Basic Functions

\[ L\{1\} = \frac{1}{s} \]

\[ L\{t^n\} = \frac{n!}{s^{n+1}}, n = 1, 2, 3, \ldots \]

\[ L\{e^{at}\} = \frac{1}{s - a} \]

\[ L\{\sin kt\} = \frac{k}{s^2 + k^2} \]

\[ L\{\cos kt\} = \frac{s}{s^2 + k^2} \]

\[ L\{\sinh kt\} = \frac{k}{s^2 - k^2} \]

\[ L\{\cosh kt\} = \frac{s}{s^2 - k^2} \]
Find $\int \sin kt^2 \, dt$

$\int \sin kt^2 \, dt = \int_0^\infty e^{-st} \sin kt \, dt$ (by parts)

$\lim_{b \to \infty} \left[ \left( -\frac{k \cos(kt)}{s^2 + k^2} - \frac{\sin(kt) \cdot s}{s^2 + k^2} \right) e^{st} \right]_0^b$

$= \lim_{b \to \infty} \left( \frac{-k \cos(kt) - \sin(kt) \cdot s}{s^2 + k^2} \right) e^{st} - 0$

$= \frac{k}{s^2 + k^2} \quad s > 0$
Definitions:

A function $f$ is **piecewise continuous** on $[0, \infty)$ if, in any interval $0 \leq a \leq t \leq b$ there are at most a finite number of points $t_k, k = 1, 2, \ldots, n, (t_{k-1} < t_k)$ at which $f$ has finite discontinuities and is continuous on each open interval $t_{k-1} < t < t_k$.

![Graphs showing piecewise continuous and not piecewise continuous functions]
A function $f$ is said to be of exponential order if there exist numbers $c, M > 0, T > 0$ such that

$$|f(t)| \leq Me^{cT}, t > T.$$
Any positive integral power of $t$ is of exponential order for $c>0$:

$$|t^n| \leq M \cdot e^{ct}, \quad \left| \frac{t^n}{e^{ct}} \right| \leq M, t > T.$$  

(Apply L'Hopital's Rule $n$ times.)

**Theorem 5.2  Sufficient Conditions for Existence**

If $f(t)$ is piecewise continuous on the interval $[0, \infty)$ and of exponential order for $t>T$, then $L\{f(t)\}$ exists for $s>c$.

**Theorem 5.3  Behavior of $F(s)$ as $s \to \infty$**

If $f$ is piecewise continuous on $[0, \infty)$ and of exponential order for $t>T$, then $\lim_{s \to \infty} L\{f(t)\} = 0$.

$L\{\sin kt\} = \frac{k}{s^2+k^2}$

$\sin(kt)$ exists for $s>0$ and is of exponential order for $t \geq 0$.  

$\frac{\text{continuous}}{\text{sin(kt) is of exponential order}}$