Section 2.2 (Homework)

#13. \((e^y+1)^2 - e^y \, dt + (e^t+1)^3 \, e^{-t} \, dy = 0\)

(mult. by \(t\))

\[ \frac{e^t}{(e^t+1)^2} \, dt + \frac{e^y}{(e^y+1)^3} \, dy = 0 \]

(integrate)

\[ \int (e^t+1)^{-3} \, e^t \, dt + \int (e^y+1)^{-2} \, e^y \, dy = 0 \]

\[ \int u^{-2} \, du + \int v^{-1} \, dv = 0 \]

\[ \frac{u^{-2}}{-2} + \frac{v^{-1}}{-1} = C_1 \]

\[ \frac{-1}{2(e^t+1)^2} - \frac{1}{e^y+1} = C_1 \]

(mult. by 2)

\[ \frac{1}{2(e^t+1)^2} + \frac{1}{e^y+1} = C_2 \]

\[ \frac{1}{(e^t+1)^2} + \frac{2}{e^y+1} = C \]
#17. \[
\frac{dp}{dt} = p - p^2 \quad (\text{mult. by } \frac{1}{p-p^2} dt)
\]
\[
\frac{dp}{p(1-p)} = dt
\]
\[
\frac{dp}{p} - \frac{dp}{p-1} = dt \quad \text{(integrate)}
\]
\[
\ln|p| - \ln|p-1| = t + c,
\]
\[
\ln\left|\frac{p}{p-1}\right| = t + c,
\]
\[
\frac{p}{p-1} = e^{t+c} = e^{t+c^2}
\]
\[
\frac{p}{p-1} = c_2 e^t \quad (\text{mult. by } p-1)
\]
\[
p = \frac{c_2 e^t}{1 - c_2 e^t} \quad 1 - c_2 e^t = p \frac{c_2 e^t}{p(c_2 e^t - 1)} \Rightarrow
\]
\[
p = \frac{c_2 e^t}{c_2 e^t - 1} \quad \frac{1}{-1}
\]
\[
-c_2 = c
\]
\[
\rho = \frac{ce^t}{1 + ce^t}
\]
3. Find a continuous solution satisfying
\[
\frac{dy}{dt} + y = f(t)
\]
\[
\begin{cases}
0, & 0 \leq t \leq 1 \\
1, & t > 1
\end{cases}
\]
\[
f(t) = \begin{cases}
-1, & t > 1
\end{cases}, \quad y(0) = 1
\]

Note: Check continuity of solution at \( t = 1 \)

\[
p(t) = 1 \quad e^{\int p(t)\,dt} = e^{\int 0\,dt} = e^t = e^t (e^{\int f(t)\,dt})
\]

\[
\begin{cases}
- e^t, & 0 \leq t \leq 1 \\
 e^t & t > 1
\end{cases}
\]

\[
d[e^t y] = \begin{cases}
- e^t & 0 \leq t \leq 1 \\
 e^t & t > 1
\end{cases} \quad \text{(integrate)}
\]

\[
e^t y = \begin{cases}
 e^t + c_1, & 0 \leq t \leq 1 \\
 - e^t + c_2 & t > 1
\end{cases}
\]

\[
y(0) = 1 \quad e^0 = 1 + c_1 \quad \iff \quad 1 = 1 + c_1, \quad c_1 = 0
\]

\[
\Rightarrow \quad e^t y = \begin{cases}
 e^t, & 0 \leq t \leq 1 \\
 - e^t + c_2 & t > 1
\end{cases}
\]
\[ y(0) = 1 \quad e^0(1) = e^0 + c_1 \iff 1 = 1 + c_1 \quad \therefore c_1 = 0 \]

\[ e^t y = \begin{cases} 
1 & 0 \leq t \leq 1 \\
-e^t + c_2 & t > 1 
\end{cases} \]

To be continuous at \( t = 1 \), we must have \( e = -e + c_2 \)

\[ \therefore e = -e + c_2 \iff c_2 = 2e \]

\[ e^t y = \begin{cases} 
1 & 0 \leq t \leq 1 \\
-e^t + 2e & t > 1 
\end{cases} \]

\[ y = \begin{cases} 
1 & 0 \leq t \leq 1 \\
-1 + 2e^{1-t} & t > 1 
\end{cases} \]
Omit Functions defined by Integrals and Example 6 in text.
Substitutions: Bernoulli’s Equation

The DE $\frac{dy}{dt} + P(t)y = f(t)y^n$ where $n$ is any real number is called a Bernoulli equation. For $n \neq 0, n \neq 1$ the substitution $u = y^{1-n}$ reduces any nonlinear Bernoulli equation to a linear one.

\[ n = 0 \quad \frac{dy}{dt} + P(t)y = f(t) \quad \text{(Linear)} \]

\[ n = 1 \quad \frac{dy}{dt} + P(t)y = f(t)y \]

\[ \frac{dy}{dt} + [P(t) - f(t)]y = 0 \]

\[ \iff \frac{dy}{y} + [P(t) - f(t)]dt = 0 \quad \text{(variables are separated)} \]
Example: Find the general solution of the DE.

\[
\ast \quad \frac{dy}{dt} - y = e^t y^2 \quad \text{(Bernoulli)}
\]

\[
u = \frac{y}{y^2} \Rightarrow y = \frac{1}{u}
\]

4. \quad u = y^{1-n} = y^{-2} = y^{-1} \Rightarrow y = u^{-1} = \frac{1}{u}

\[
\frac{dy}{dt} = -\frac{1}{u^2} \frac{du}{dt}
\]

\[
-\frac{1}{u^2} \frac{du}{dt} - \frac{1}{u} = e^t \frac{1}{u^2} \quad (\text{mult. by } u^2)
\]

\[
\frac{du}{dt} + u = -e^t \quad \text{(Linear)}
\]

\[
\exp \left( \int \frac{1}{u} \, dt \right) e^t = e^{\int \frac{1}{u} \, dt} = e^t
\]  

(mult. by \( e^t \, dt \))

\[
e^t du + u e^t \, dt = -e^t \, dt
\]

\[
d[e^t u] = -e^t \, dt \quad \text{(integrate)}
\]

(mult. by \( e^t \))

\[
e^t u = -\frac{1}{2} e^t + c_1
\]

\[
u = -\frac{1}{2} e^t + c_1 e^{-t}
\]

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\[ u = -\frac{1}{2} e^t + c_1 e^{-t} \]

\[ \frac{1}{y} = -\frac{1}{2} e^t + c_1 e^{-t} \quad \iff \quad \frac{1}{y} = -\frac{1}{2} e^t + c_1 e^{-t} \]

\[ y = \frac{-2}{e^t + c e^{-t}} \]

\[ -2c_1 = c \]
2.4 Mathematical Models

The mathematical description of a system or a phenomenon is called a mathematical model.

A number of variables may be responsible for changing the system. Choosing which variables to use for a model is called determining the level of resolution of the model.

A mathematical model of a physical system often involves the variable time $t$. A solution of the model is said to give the state of the system.
A mathematical model consists of

1. Identifying variables responsible for changing the system.
2. A set of reasonable assumptions about the system.
3. A mathematical formulation which is often a differential equation or a system of differential equations.
4. The solution which gives the state of the system.
Linear models

\[ \frac{dx}{dt} = kx, \quad x(t_0) = x_0, \quad k \text{ a proportionality constant, has} \]
a solution of the form \( x = x_0 e^{kt} \) or \( x = x_0 a^t \).

\[ \frac{dx}{dt} = kx \quad \text{(mult. by } \frac{1}{x} \text{ dt)} \]

\[ \frac{dx}{x} = k \, dt \quad \text{(integrate)} \]

\[ \ln|\!\!x\!\!| = kt + c_1 \quad \iff \quad x = c_2 e^{kt} \]

\[ e^{\ln|x|} = e^{kt + c_1} = e^{kt} e^{c_1} = e^{c_2} \]

\[ x = c_2 e^{kt} \quad \text{if } t = 0, \quad x(0) = x_0 \]

\[ x(0) = x_0 = c_2 e^{c_2} \quad \iff \quad x_0 = c_2 \]

\[ x = x_0 e^{kt} \]

\[ x_0 e^{kt} = x_0 e^{kt} \quad \Rightarrow \quad x = x_0 e^{kt} \]

\[ x = x_0 a^t \]

\[ x_0 e^{kt} = x_0 (e^k)^t \quad \Rightarrow \quad x = x_0 a^t \]
Example (from text): Carbon Dating
Isotope carbon 14 is produced in the atmosphere by the action of cosmic radiation on nitrogen. The ratio of C-14 to ordinary carbon in the atmosphere, and thus in all living things, appears to be constant. When an organism dies, the absorption of C-14 by eating or breathing ceases. By comparing the proportionate amount of C-14 in a fossil to the amount in the atmosphere, the age of the fossil can be estimated. Carbon dating is thus based on the radioactive decay of C-14. If $A = \text{amount of C-14 present at time } t$, $A(5600) = \frac{A_0}{2}$. Thus, the half-life of C-14 is 5600 years.
\[ A = A_0 e^{kt} \]

\[ \frac{A_0}{2} = A_0 e^{k(5600)} \Rightarrow \frac{1}{2} = e^{5600k} \]

\[ \ln\left(\frac{1}{2}\right) = 5600k \Rightarrow k = \frac{1}{5600}(-\ln 2) = -.00012378 \]

\[ A = A_0 e^{-0.00012378t}, t \leq 50000 \]

(After 50000 years, too much may need to be destroyed and it may be too hard to distinguish from background radiation.)
Example
1. A fossilized bone is found to contain 1/100 of the original amount of C-14. Determine its age.

\[ A = \frac{A_0}{100} = A_0 e^{-0.00012378t} \Rightarrow \]

\[ \frac{1}{100} = e^{-0.00012378t} \Rightarrow -\ln 100 = -0.00012378t \Rightarrow \]

\[ t = \frac{\ln 100}{0.00012378} = 37204.48 \]

or approximately 372000 years old.
Nonlinear Models

If \( P = \text{population} \), \( \frac{dP}{dt} = kP \) is one model for population change. In this model, the specific or relative growth rate is defined by \( k = \frac{dP/dt}{P} \) [rate of change per person]
A better model was found by Belgian mathematician-biologist P.F. Verhulst who studied U.S. population from 1790 – 1840. He used the model

\[
\frac{dP}{dt} = P(a - bP), \, a > 0, b > 0.
\]

This is an example of a density-dependent hypothesis.

One way to look at this model is as follows:

\[
\frac{1}{P} \frac{dP}{dt} = \text{rate of growth per individual in a population}
\]

\[
\frac{1}{P} \frac{dP}{dt} = (\text{average birth rate}) - (\text{average death rate})
\]

\[
= a - bP
\]

\[
\frac{dP}{dt} = P(a - bP) \quad \text{This is the logistic equation.}
\]

The solution to the logistic equation is the logistic function and the graph of the logistic function is a logistic curve.
To solve the logistic equation, first separate variables:

\[
\frac{dP}{P(a-bP)} = dt
\]

\[
\frac{(1/a)dP}{P} + \frac{(b/a)dP}{a-bP} = dt
\]

\[
\frac{(1/a)dP}{P} - \frac{1}{a} \frac{(-bdP)}{a-bP} = dt
\]

assumed $P$ is growing

\[
\frac{1}{a} \ln P - \frac{1}{a} \ln(a-bP) = t + c_1
\]

\[
\ln \left( \frac{P}{a-bP} \right) = at + c_2
\]

\[
e^{\ln \left( \frac{P}{a-bP} \right)} = e^{at+c_2} = e^{at} e^{c_2}
\]

\[
\frac{P}{a-bP} = c_3 e^{at}
\]

$P(0)=P_0$

\[
P_0 = \frac{ac}{bc+1}
\]

\[
p'o = \frac{ac}{bc+1}
\]

\[
-P'o = -\frac{ac}{bc+1}
\]

\[
\frac{P_0}{bc+1} = \frac{(a-P_0)b}{c}
\]

\[
\Rightarrow c = \frac{P_0}{a-P_0 b}
\]
Integrate both sides and simplify. The text gives the solution in the form

\[ P(t) = \frac{aP_0 e^{at}}{a - bP_0 + bP_0 e^{at}} = \frac{aP_0}{(a - bP_0)e^{-at} + bP_0} \]

The calculator gives a different form:
Verhulst’s model for the U.S. population was
\[
\frac{dP}{dt} = P(0.0314 - 0.000159P) \\
P(t) = \frac{770.2}{3.9 + 193.6e^{-0.0314t}}
\]

, with \( t = 0 \) in 1790.

Examples:

2. Use the model to estimate the population in 1840.
   \( P(50) = 17.4 \text{ million} \quad (\text{actual} = 17.1 \text{ million}) \)

3. Use the model to estimate the population in 1890.
   \( P(100) = 62.7 \text{ million} \quad (\text{actual} = 63.0 \text{ million}) \)
4. Use the model to estimate the population in 1990.

\[ P(200) = 180.7 \text{ million} \quad (\text{actual} = 249 \text{ million}) \]

5. Find the limiting population:

\[
\lim_{t \to \infty} P(t) = \lim_{t \to \infty} \frac{770.2}{3.9 + 193.6e^{-0.0314t}} = \frac{770.2}{3.9} = 197.5
\]
Here: \( \frac{dP}{dt} = P(0.0314 - 0.000159P) \)

\( t = 0 \) corresponds to 1840 

\( P(0) = 17.1 \) (actually 197.40)

\( \lim_{t \to \infty} P > 180.7 \)

\[
P = \frac{(0.0314t)}{0.00159 + 0.0314t}
\]

\( P(0) = 17.1 \) 

\( 17.1 = \frac{(0.0314)(0)}{0.000159 + 0.0314(0) + 0.0314(0.0534)} \)

\( c = 0.0534/6 \)

\( \Rightarrow P = \frac{0.0314t}{0.000159 + 0.0314t + 0.0314(0.0534/6)} \)

\( \Rightarrow P = \frac{0.0314}{0.000159 + 0.0314(0.0534/6) - 0.0314t} \)

\( \Rightarrow \lim_{t \to \infty} P = \frac{0.0314}{0.000159} = 197. \)
Falling bodies and air resistance

A falling body of mass $m$ encounters air resistance proportional to instantaneous velocity.

\[ \begin{array}{c}
\downarrow & -kv \\
\hline & mg \\
\uparrow & +
\end{array} \]

The net force is $mg - kv$. Using Newton’s second law, we have the mathematical formulation:

\[ m \frac{dv}{dt} = mg - kv. \]

A second model, assuming high-speed motion (such as a sky diver before his parachute opens) is given by:

\[ m \frac{dv}{dt} = mg - kv^2. \]
Example:

6. Solve the DE \( \frac{dv}{dt} = mg - kv \) subject to \( v(0) = v_0 \).

\[
m \frac{dv}{dt} = mg - kv
\]

\[
\frac{dv}{dt} = g - \frac{k}{m} v \Rightarrow \frac{dv}{g - \frac{k}{m} v} = dt
\]

\[
-u = g - \frac{k}{m} v \quad du = -\frac{k}{m} dv
\]

\[
\ln \left| g - \frac{k}{m} v \right| = -\frac{k}{m} t + c_1
\]

\[
\frac{k}{m} v = c e^{-\frac{kt}{m}}
\]

\[
\left( g - \frac{k}{m} v \right) = \left( g - \frac{k}{m} v_0 \right) e^{-\frac{kt}{m}}
\]

\[
v(0) = v_0 \Rightarrow g - \frac{k}{m} v_0 = c \Rightarrow ...
\]

\[
v(t) = \frac{mg}{k} + (v_0 - \frac{mg}{k}) e^{-\frac{kt}{m}}
\]

The terminal velocity is \( \lim_{t \to \infty} v(t) = \frac{mg}{k} \).
Homework Sections 2.3 and 2.4