1. (12%) State whether the given differential equation is linear or non-linear and give the order of the equation.

<table>
<thead>
<tr>
<th>Linear or Non-linear</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>a. ( x^2 \left( \frac{dy}{dx} \right)^2 + x \left( \frac{dy}{dx} \right) + 1 = 0 )</td>
<td>Non-linear 1st</td>
</tr>
<tr>
<td>b. ( \frac{dy}{dx} + 2x \sqrt{y-1} )</td>
<td>Non-linear 1st</td>
</tr>
<tr>
<td>c. ( \frac{dx}{dt} + x \cdot e^2 = 1 )</td>
<td>Non-linear 1st</td>
</tr>
</tbody>
</table>

\[ x \frac{dx}{dt} + t^2 = \sin t \quad \text{Non-linear 1st} \]
\[ t^2 \frac{d^2 x}{dt^2} + e^x \frac{dx}{dt} = e^t \quad \text{Linear and} \]
2. (12%) Verify that \( y = 3\cos 2\theta + 5\sin 2\theta \) is a solution of the differential equation \( y'' + 4y = 0 \).

\[
\begin{align*}
y' &= -6\sin 2\theta + 10\cos 2\theta \\
y'' &= -12\cos 2\theta - 20\sin 2\theta \\
+y &= 12\cos 2\theta + 20\sin 2\theta \\
y'' + 4y &= 0 \checkmark
\end{align*}
\]
3. Use the fact that \( y = c_1 x^2 + c_2 x^{-3} \) is a two-parameter family of solutions of \( x^2 y'' - xy' + y = 0 \) to find a solution with initial conditions \( y(1) = 5 \) and \( y'(1) = 10 \).

\[
y = c_1 x^2 + c_2 x^{-3}
\]

\( y(1) = 5 \);

\( y'(1) = 10 \).

\[
y = c_1 + \frac{c_2}{x^3}
\]

\( \frac{dy}{dx} = 2c_1 x - 3c_2 x^{-4} \)

\( y'(1) = 10 \);

\( 10 = 2c_1 - 3c_2 \).

\[
5 = c_1 + c_2 \quad \text{(mul. by 3)}
\]

\( 15 = 3c_1 + 3c_2 \);

\( 10 = 2c_1 - 3c_2 \).

\( 25 = 5c_1 \).

\( 5 = c_1 \).

\( 5 = c_2 \).

\[
y = 5x^2
\]
4. (18%) Given the differential equation \( \frac{dx}{dt} = x(1 - 2x) \).

   a. Use your calculator to find the slope field for the differential equation using tStep = .5 and use the Decimal screen. Sketch below.

   ![Slope field diagram]

   b. On the diagram above, sketch the solution with initial condition \( x(-5) = .2 \).
c. Find the critical points of the DE and identify each as asymptotically stable or unstable.

\[ \frac{dx}{dt} = x(1 - 0.2x) = 0 \]

\[ x = 0 \quad \text{unstable} \]

\[ 1 - 0.2x = 0 \quad x = 5 \quad \text{asymptotically stable} \]

d. Sketch the phase portrait for this DE.

<table>
<thead>
<tr>
<th>int.</th>
<th>sign of ( \frac{dx}{dt} )</th>
<th>( x(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (-\infty, 0) )</td>
<td>-</td>
<td>↓</td>
</tr>
<tr>
<td>( (0, 5) )</td>
<td>+</td>
<td>↑</td>
</tr>
<tr>
<td>( (5, \infty) )</td>
<td>-</td>
<td>↓</td>
</tr>
</tbody>
</table>
5. (12%) Solve the following differential equations.

a. \( \sin x \sin y + \cos x \cos y \frac{dy}{dx} = 0 \)

\[
\begin{align*}
\frac{\sin x}{\cos x} \frac{dx}{dy} + \frac{\cos y}{\sin y} \frac{dy}{dx} &= 0 \\
\tan x \, dx + \cot y \, dy &= 0 \\
-\ln |\cos x| + \ln |\sin y| &= \ln c, \\
\frac{\sin y}{\cos x} &= c \\
\frac{\sin y}{\cos x} &= c \\
\sin y &= c \cos x \\
\text{or } y &= \arcsin(c \cos x)
\end{align*}
\]
b. \(\frac{dy}{dx} + \frac{1}{x}y = e^x\)

\(P(x)\) (mult. by \(x\)dx)

\[x \frac{dy}{dx} + y \, dx = xe^x \, dx\]

\[\int [xy] = xe^x \, dx\] (integrate)

\[xy = \int xe^x \, dx + c,\]

\[xy = (x-1)e^x + c,\]

\(y = (1 - \frac{1}{x})e^x + \frac{c}{x}\) (mult. by \(\frac{1}{x}\))
b. \((1 + x)y' + y = \cos(x)\)

First-order linear

(mult. by \(\frac{1}{1+x}\))

\[
y' + \left(\frac{1}{1+x}\right)y = \frac{1}{1+x} \cos(x)
\]

\[
y' + P(x)y = \frac{1}{1+x} \cos(x)
\]

\[
\int P(x) \, dx = \int \frac{1}{1+x} \, dx = \ln(1+x)
\]

* (mult. by \((1+x) \, dx\))

\[
(1+x)y' + y \, dx = \cos(x) \, dx
\]

\[
\int [(1+x) \, y] = \cos(x) \, dx \quad {\text{(integrate)}}
\]

\[
(1+x) \, y = \sin(x) + C
\]

\[
y = \frac{\sin(x) + C}{1+x}
\]
c. \( x^2y' = xy + y^2 \) \[ \frac{d^2y}{dx^2} = xy + y^2 \]

\[ \text{mult: by } dx \]

\[ x^2 \frac{dy}{dx} = xy \, dx + y^2 \, dx \]

\[ -x^2 \frac{dy}{dx} = -x^2 \frac{dy}{dx} \]

\[ 0 = (xy + y^2) \, dx - x^2 \, dy \]

\[ M(x,y) \quad N(x,y) \]

\[ M(kx_1, ky_1) = (kx)(ky) + (ky) \cdot k^2 (xy + y^2) \]

\[ = k^2 M(x, y) \]

homogeneous of degree 2

\[ N(kx_1, ky_1) = -(kx) \cdot k^2 \cdot k^2 (-x^2) \]

\[ = k^2 N(x, y) \]

homogeneous of degree 2

Let \( y = ux \)
C. Let \( y = ux \)
\[
\frac{dy}{dx} = u \frac{dx}{dx} + x \frac{du}{dx}
\]
\[
x^2 \frac{dy}{dx} = xy \frac{dx}{dx} + y \frac{dx}{dx}
\]
\[
x^2 (udx + xdu) = x (ux) \frac{dx}{dx} + (ux)^2 \frac{dx}{dx}
\]
\[
x^2 dx + x^2 du = ux^2 \frac{dx}{dx} + u^2 x^2 \frac{dx}{dx}
\]
\[
-ux^2 \frac{dx}{dx} - u \frac{dx}{dx}
\]
\[
x^2 du = u \frac{dx}{dx} \quad \text{(multiply by } \frac{1}{x^2})
\]
\[
\frac{du}{u^2} = \frac{dx}{x} \quad \text{(integrate)}
\]
\[
\frac{-1}{u} = \ln |x| + C_1, \quad \frac{y}{u} = ux
\]
\[
\frac{-1}{u} = \ln |x| + C_1, \quad \frac{\frac{dy}{dx}}{x} = u
\]
\[
\frac{x}{y} = \ln |x| + C_1, \quad \frac{1}{u} = \frac{1}{u}
\]
\[
take \quad \text{(reciprocals)}
\]
\[
\frac{-\frac{dy}{dx}}{x} = \frac{1}{\ln |x| + C_1}
\]
\[
\frac{dy}{dx} = -\frac{x}{\ln |x| + C_1}
\]
\[
y = \frac{x}{c_2 - \ln |x|}
\]
6. (12%) When interest is compounded continuously, the amount of money increases at a rate proportional to the amount $S$ present at time $t$. 

\[ \frac{dS}{dt} = rs \]

where $r$ is the annual rate of interest expressed as a decimal.

a. If the annual rate of interest is 4.5% in how many years will an initial deposit double? 

\[ r = 0.045 \]

\[ S = S_0 e^{rt} \]

Find $t$ such that \[ S = 2S_0 \]

\[ 2S_0 = S_0 e^{rt} \]

\[ 2 = e^{rt} \]

\[ \ln 2 = \ln e^{rt} \]

\[ \ln 2 = 0.045t \]

\[ t = \frac{\ln 2}{0.045} = 15.4 \text{ years} \]
b. If half the interest is withdrawn after the first year, find the amount of money accrued by the end of the second year.

When \( t = 1 \), \( S = S_0 e^{.045(t)} \)

**Interest earned after 1 year**

\[
\frac{1}{2} \text{ first year interest} = \frac{S_0 e^{.045} - S_0}{2} = \frac{S_0 (e^{.045} - 1)}{2}
\]

If \( S_1 \) = amount present after 1st year

\[
S_1 = S_0 e^{.045} \left[ \frac{S_0 (e^{.045} - 1)}{2} \right]
\]

\[
= \frac{S_0 e^{.045}}{2} + \frac{S_0}{2} = \frac{S_0 (e^{.045} + 1)}{2}
\]

Am't. reinvested after 1 year

After 2 years

\[
S = \left[ \frac{S_0 (e^{.045} + 1)}{2} \right] e^{.045(t)}
\]

\[
= \frac{S_0 (e^{.045} + 1) e^{.045}}{2}
\]

\[
\approx 1.0701 S_0
\]
3.1 Linear Equations

Initial Value Problem for and $n$th order Linear Equation:

Solve:

$$a_n(t)\frac{d^n y}{dt^n} + a_{n-1}(t)\frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1(t)\frac{dy}{dt} + a_0(t)y = g(t)$$

Subject to:

$$y(t_0) = y_0; \quad y'(t_0) = y_1; \quad \cdots; \quad y^{n-1}(t_0) = y_{n-1}$$

(initial conditions)
Theorem 3.1 (Existence of a Unique Solution) Let $a_n(t), a_{n-1}(t), \ldots, a_0(t), g(t)$ be continuous on an interval $I$ and $a_n(t)$ not 0 for every $t$ in $I$. If $t = t_0$ is any point in this interval, then a solution $y(t)$ of the IVP exists on the interval and is unique.
Example:

1. Find an interval around $t=0$ for which the given initial value problem has a unique solution if $(t-2)y''+3y=t, y(0)=0; y'(0)=1.$

Interval: $t-2 \text{ is not } 0$, $t \text{ is not } 2$

$(-\infty, 2)$
Boundary Value Problem (BVP)

Solve:
\[ a_n(t) \frac{d^n y}{dt^n} + a_{n-1}(t) \frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_1(t) \frac{dy}{dt} + a_0(t) y = g(t) \]

Subject to: \[ y(a_1) = y_0; y(a_2) = y_1; \ldots; y(a_n) = y_{n-1} \text{ or} \]
\[ y(a_1) = y_0; y'(a_2) = y_1; \ldots; y^{n-1}(a_n) = y_{n-1} \text{ etc.} \]

These are all special cases of the system:

\[ m_1 y(a_1) + n_1 y'(a_1) + \ldots + \ldots = r_1 \]
\[ m_2 y(a_2) + n_2 y'(a_2) + \ldots + \ldots = r_2 \]
\[ \ldots \]
\[ m_n y(a_1) + n_n y'(a_1) + \ldots + \ldots = r_n \]

A BVP can have many, one, or no solutions.
Example 3 from page 120: For the DE $x'' + 16x = 0$,

$x = c_1 \cos 4t + c_2 \sin 4t$ is the two parameter family of solutions. Consider the boundary value problems with:

a. $x(0) = 0, x(\pi/2) = 0$

   $x(0) = 0 = c_1 + 0 \Rightarrow c_1 = 0$
   $x(\pi/2) = 0 = c_1 + 0 \Rightarrow c_1 = 0$

   many solutions

   $c_2$ can be any number

b. $x(0) = 0, x(\pi/8) = 0$

   $x(0) = 0 = c_1 + 0 \Rightarrow c_1 = 0$
   $x(\pi/8) = c_1 \cos \pi/2 + c_2 \sin \pi/2 \Rightarrow 0 = 0 + c_2 \Rightarrow c_2 = 0$

   one solution

   $x = 0$

c. $x(0) = 0, x(\pi/2) = 1$

   $x(0) = 0 = c_1 + 0 \Rightarrow c_1 = 0$

   $x(\pi/2) = 1 = c_1 \cos 2\pi + c_2 \sin 2\pi$

   $1 = c_1 + 0 \Rightarrow c_1 = 1$

   no solution

   inconsistent
Example 2:

Given that \( y = c_1 t^2 + c_2 t^4 + 3 \) is a two parameter family of solutions of the DE \( t^2 y'' - 5ty' + 8y = 24 \) for all real numbers, determine whether a member of the family can be found that satisfies the boundary conditions:

b. \( y(0) = 1 \) and \( y(1) = 2 \)

\( y(0) = 0 + 0 + 3 = 3 \) \( \Rightarrow y(0) = 3 \)

\( y(0) = 1 = 3 \) ?? impossible; no solution

c. \( y(0) = 3 \) and \( y(1) = 0 \) (many solutions)

\( y(0) = c_1 + c_2 + 3 = 0 \) (no unique soln.)

d. \( y(1) = 3 \) and \( y(2) = 15 \) (one solution)

\( y(1) = c_1 + c_2 + 3 = 3 \) \( \Rightarrow c_1 + c_2 = 0 \) (multi. soln.)

\( y(2) = c_1(4) + c_2(16) + 3 = 15 \)

\( 4c_1 + 16c_2 = 12 \)

\( c_1 + 4c_2 = 3 \)

\( -c_1 - c_2 = 0 \)

\( \frac{c_1 + 4c_2}{3c_2} = 3 \) \( \Rightarrow c_2 = 1 \)

\( c_1 = -1 \) unique soln.