Section 3.4 (Homework)

#33. \( y'' + y = \cos^2 t \)

aux. eqn. \( m^2 + 1 = 0 \) \( m = \pm i \)
\( y_c = c_1 \cos t + c_2 \sin t \)

assume: \( y_p = u_1 y_1 + u_2 y_2 \)
\( y_p = u_1 \cos t + u_2 \sin t \)

using variation of parameters, this reduces to solving:
\[ u_1 y_1' + u_2 y_2' = 0 \]
\[ u_1' y_1 + u_2' y_2 = f(t) \]
\[ u_1' \cos t + u_2' \sin t = 0 \]
\[ u_1 y_1' + u_2 y_2' = f(t) - u_1' \sin t + u_2' \cos t = \cos^2 t \]
\[
\begin{align*}
    u_1' \cos t + u_2' \sin t &= 0 \\
    -u_1' \sin t + u_2' \cos t &= \cos^2 t \\
    \frac{u_1'}{w_1} &= \frac{w_1}{w} \\
    u_2' &= \frac{w_2}{w} \\
    w &= \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = \cos^2 t + \sin^2 t = 1 \\
    w_1 &= \begin{vmatrix} 0 & \sin t \\ \cos^2 t & \cos t \end{vmatrix} = 0 - \cos^2 t \sin t \\
    u_1 &= \frac{w_1}{w} = -\frac{\cos^2 t \sin t}{1} \\
    u_1 &= \int \cos^2 t (-\sin t) dt = \int u^2 du = \frac{u^3}{3} = \frac{\cos^3 t}{3} = u_1
\end{align*}
\]
\[ w_2 = \begin{vmatrix} \cos t & 0 \\ -\sin t & \cos^2 t \end{vmatrix} = \cos^3 t - 0 \]

\[ u_2 = \frac{w_2}{W} = \frac{\cos^3 t}{1} = \cos^3 t \]

\[ u_2 = \int \cos^3 t \, dt = \int (1 - \sin^2 t) \cos t \, dt \]

\[ u = \sin t, \quad du = \cos t \, dt \]

\[ = \int (1 - u^2) \, du = u - \frac{u^3}{3} \]

\[ u_2 = \sin t - \frac{\sin^3 t}{3} \]

\[ y = y_c + y_p = y_c + u_1 y_1 + u_2 y_2 \]

\[ y = c_1 \cos t + c_2 \sin t + \frac{\cos^3 t}{3} (\cos t) \]

\[ + (\sin t - \frac{\sin^3 t}{3}) \sin t \]

\[ = c_1 \cos t + c_2 \sin t + \frac{\cos^4 t}{3} + \sin^2 t \]

\[ - \frac{\sin^4 t}{3} \]
\[ y = c_1 \cos t + c_2 \sin t + \sin^2 t \]
\[ + \frac{1}{3} [\cos^4 t - \sin^4 t] \]
\[ = c_1 \cos t + c_2 \sin t + \sin^2 t \]
\[ + \frac{1}{3} (\cos^2 t - \sin^2 t)(\cos^2 t + \sin^2 t) \]
\[ = c_1 \cos t + c_2 \sin t + \sin^2 t \]
\[ + \frac{1}{3} (\cos^2 t - \sin^2 t) \]
#11. \( t^2 y'' + 5ty' + 4y = 0 \)

(Cauchy-Euler)

Assume solution: \( y = t^m \); \( y' = mt^{m-1} \);

\[ y'' = m(m-1)t^{m-2} \]

\[ t^2 \left[ m(m-1)t^{m-2} + 5mt^{m-1} + 4t^m \right] = 0 \]

\[ t^m \left[ m(m-1) + 5m + 4 \right] = 0 \]

Auxiliary eqn:

\[ m^2 - m + 5m + 4 = 0 \]

\[ m^2 + 4m + 4 = 0 \]

\( (m+2)^2 = 0 \)

\( m = -2, -2 \)

\[ y = c_1 t^{2} + c_2 t^{2} \ln t \]
\#21 \quad \frac{t^4}{4} \frac{d^4 y}{dt^4} + \frac{6}{3} \frac{d^3 y}{dt^3} = 0 \quad \text{(by \( t^3 \))}

\text{(Cauchy)}

\frac{t^4}{4} \frac{d^4 y}{dt^4} + \frac{6}{3} \frac{d^3 y}{dt^3} = 0 \quad \text{(-Euler)}

\begin{align*}
y &= t^m, \quad y' = mt^{m-1}, \quad y'' = m(m-1)t^{m-2},
\end{align*}

\begin{align*}
y''' &= m(m-1)(m-2)t^{m-3}, \quad y'''' = m(m-1)(m-2)(m-3)t^{m-4},
\end{align*}

\begin{align*}
t^4 \left[ m(m-1)(m-2)(m-3)t^{m-4} \right] + 6t^3 \left[ m(m-1)(m-2)t^{m-3} \right] &= 0
\end{align*}

\begin{align*}
t^m(m)(m-1)(m-2) \left[ m-3 + 6 \right] &= 0
\end{align*}

\text{aux. eqn.} \quad m(m-1)(m-2)(m+3) = 0

m = 0, 1, 2, -3

\begin{align*}
y &= c_1 + c_2t + c_3t^2 + c_4t^3
\end{align*}
3.6 Mathematical Models: Initial-Value Problems

A flexible spring is suspended vertically from a rigid support and a mass \( m \) is attached to its end. By Hooke’s Law, the spring exerts a restoring force \( F \) opposite to the direction of elongation and proportional to the amount of elongation \( s \); that is, \( F = ks \), where \( k \) is a constant of proportionality called the spring constant. At equilibrium, the restoring force balances the weight \( W \) of the mass, where \( W = mg \), so that \( mg - ks = 0 \). Remember that \( g \) is the acceleration due to gravity, \( g = 32 \frac{ft}{sec^2} = 980 \frac{cm}{sec^2} \) and mass \( m \) is measured in slugs, kilograms, or grams.
If the mass is displaced from equilibrium, the restoring force is $k(x + s)$. With no retarding forces or external forces (free motion) and using Newton’s Second Law, $F = ma$, we have

$$m \frac{d^2x}{dt^2} = -k(s + x) + mg$$

$$m \frac{d^2x}{dt^2} = -ks - kx + mg = mg - ks - kx = -kx$$

$$m \frac{d^2x}{dt^2} = -kx$$

$$m \frac{d^2x}{dt^2} + kx = 0$$

$$\frac{d^2x}{dt^2} + \frac{k}{m} x = 0$$

$$\frac{d^2x}{dt^2} + \omega^2 x = 0, \omega^2 = \frac{k}{m}$$
This equation describes simple harmonic motion or free undamped motion with initial displacement \( x(0) = \alpha \) and initial velocity \( x'(0) = \beta \).

Solutions of the auxiliary equation \( m^2 + \omega^2 = 0 \) are \( \pm \omega i \) so that the solution is given by \( x = c_1 \cos \omega t + c_2 \sin \omega t \).

**Period** \( T = \frac{2\pi}{\omega} \); **frequency** \( f = \frac{1}{T} = \frac{\omega}{2\pi} \); \( \omega \) is called the circular or angular frequency. Extreme displacement is distance as far as possible above or below equilibrium.

When constants are determined from the initial conditions, we have the **equation of motion**.
Examples: These examples are from pages 185 – 189.

#2. A 20 kg mass is attached to a spring. If the frequency of simple harmonic motion is \( \frac{2}{\pi} \) vibrations/sec., what is the spring constant \( k \)? What is the frequency of simple harmonic motion if the original mass is replaced with an 80 kg mass.

\[
\frac{f}{\pi} = \frac{2}{\pi} \quad \Rightarrow \quad 4 = \omega \quad \omega^2 = 16 = \frac{k}{m}
\]

\[
\Rightarrow \quad 16 = \frac{k}{20} \quad \Rightarrow \quad 320 = k
\]

80 kg mass \( \omega^2 = \frac{k}{m} = \frac{320}{80} = 4 \)

\[
\Rightarrow \quad \omega = 2 \cdot \frac{w}{2\pi} = \frac{2}{2\pi} = \frac{1}{\pi} \text{ vibrations/} \sec
\]
#4. A 24 lb weight attached to the end of a spring stretches it 4 inches. Find the equation of motion if the weight is released from the equilibrium position with an initial downward velocity of 2 ft./sec.

\[ F = kx \]
\[ \frac{dy}{dt} = k \left( \frac{1}{3} \right) \]
\[ \eta_2 = k \]

Convert inches to feet.
\[ 4in = \frac{1}{3} \text{ ft} \]

Convert lbs to slugs.
\[ m = \frac{W}{g} \Rightarrow m = \frac{24}{32} = \frac{3}{4} \text{ slugs} \]

\[ m \frac{d^2x}{dt^2} + kx = 0 \]

\[ 3 \frac{d^2x}{dt^2} + 72x = 0 \Rightarrow \frac{d^2x}{dt^2} + 96x = 0 \]

\[ \left[ \frac{d^2x}{dt^2} + \omega^2 x = 0 \right] \]

\[ x(0) = 0, x'(0) = 2 \ldots \]
\[ x(t) = \frac{\sqrt{6}}{12} \sin 4\sqrt{6}t \]

*Aux. Eqn.*

\[ m^2 + 96 = 0 \]
\[ m = \pm \sqrt[4]{96}i \]

\[ m = \pm \sqrt[4]{96}i \Rightarrow m = \pm 4\sqrt[4]{6}i \]

\[ x = c_1 \cos 4\sqrt{6}t + c_2 \sin 4\sqrt{6}t \]
\[ x = c_1 \cos 4\sqrt{6}t + c_2 \sin 4\sqrt{6}t \]
\[ x(0) = 0 \]
\[ x(0) = c_1 \cos (0) + c_2 \sin (0) \]
\[ 0 = c_1 + 0 \quad \Rightarrow \quad c_1 = 0 \]

\[ x = c_2 \sin 4\sqrt{6}t \]
\[ x' = 4\sqrt{6} c_2 \cos 4\sqrt{6}t \]
\[ x'(0) = 2 = 4\sqrt{6} c_2 (0) \]
\[ \frac{2}{4\sqrt{6}} = c_2 \]

\[ x = \frac{\sqrt{6}}{2\sqrt{6}} \sin 4\sqrt{6}t \quad \Rightarrow \quad x = A \sin (\omega t + \phi) \]
\[ A = \frac{\sqrt{6}}{2} \quad \Rightarrow \quad \phi = 0 \]
Alternate form of $x(t)$:

$$x(t) = A \sin(\omega t + \phi)$$

$$A = \sqrt{c_1^2 + c_2^2}; \sin \phi = \frac{c_1}{A}; \cos \phi = \frac{c_2}{A}; \tan \phi = \frac{c_1}{c_2}$$

$A$ is the amplitude and $\phi$ is the phase angle.

We have considered a model for a linear spring:

$$f(x) = kx.$$  

Other models include

- aging spring: \( F(x) = ke^{-\alpha x}, k > 0 \) and
- nonlinear spring: \( F(x) = kx + kx^3. \)
These are all called \textit{conservative systems}:

For example, the trajectories of \[ m \frac{d^2 x}{dt^2} + kx = 0 \] form a one-parameter family of solutions of

\[ m \frac{dy}{dt} + kx = 0; \quad \frac{dy}{dt} = -\frac{k}{m} x \]

\[ \frac{dy}{dt} = -\frac{k}{m} x \]

\[ \frac{dx}{dt} = \frac{y}{y} \]

\[ \frac{dy}{dx} = -\frac{kx}{my} \]

\[ my \frac{dy}{dx} + kx \frac{dx}{dy} = 0 \Rightarrow my^2 + \frac{kx^2}{2} = C \]

The trajectories are ellipses:

\[ \frac{1}{2} my^2 + \frac{1}{2} kx^2 = E \]

\text{Kinetic + Potential = Constant}

\text{Energy of mass + Energy of spring}

Therefore, the trajectories represent various energy states of the system.
Free Damped Motion

Damping forces are considered to be proportional to a power of the instantaneous velocity. Here, assume that the damping force $= \beta \frac{dx}{dt}$.

$$m \frac{d^2 x}{dt^2} = -kx - \beta \frac{dx}{dt} \quad \frac{d^2 x}{dt^2} + \frac{\beta}{m} \frac{dx}{dt} + \frac{k}{m} x = 0$$

$$\frac{d^2 x}{dt^2} + 2 \lambda \frac{dx}{dt} + \omega^2 x = 0, \quad 2 \lambda = \frac{\beta}{m}, \quad \omega^2 = \frac{k}{m}$$

The auxiliary equation is

$$m^2 + 2 \lambda m + \omega^2 = 0$$

$$m_1 = -\lambda + \sqrt{\lambda^2 - \omega^2}, \quad m_2 = -\lambda - \sqrt{\lambda^2 - \omega^2}$$
There are three cases:

1. $\lambda^2 - \omega^2 > 0$ \textbf{Overdamped} $\beta$ is large relative to $k$

\[
x(t) = c_1 e^{m_1 t} + c_2 e^{m_2 t}
\]

\[
x(t) = e^{-\lambda t} \left( c_1 e^{\sqrt{\lambda^2 - \omega^2} t} + c_2 e^{-\sqrt{\lambda^2 - \omega^2} t} \right)
\]

(no oscillatory motion)
2. \( \lambda^2 - \omega^2 = 0 \)  **Critically damped** (any slight decrease in damping force will result in oscillatory motion)

\[ x(t) = e^{-\lambda t} (c_1 + c_2 t) \]
3. \( \lambda^2 - \omega^2 < 0 \) \hspace{0.5cm} \text{Underdamped}

\[ x(t) = e^{-\lambda t} \left( c_1 \cos \sqrt{\omega^2 - \lambda^2} t + c_2 \sin \sqrt{\omega^2 - \lambda^2} t \right) \]

\[ = Ae^{-\lambda t} \sin(\sqrt{\omega^2 - \lambda^2} t + \phi) \]

\( Ae^{-\lambda t} \) is the damped amplitude

\[ \frac{2\pi}{\sqrt{\omega^2 - \lambda^2}} \] is the quasi period;

\[ \frac{\sqrt{\omega^2 - \lambda^2}}{2\pi} \] is the quasi frequency.

In any of the solutions, if \( \lim_{t \to \infty} x_i(t) = 0 \) then \( x_i(t) \) is called a transient term. If \( \lim_{t \to \infty} x_i(t) \neq 0 \) then \( x_i(t) \) is called a steady-state term.