Section 3.1 (Homework)

#23. \( y'' - 2y' + 5y = 0 \)

\[ e^{t \cos at}, e^{t \sin at} \quad (-\infty, \infty) \]

\[ W(e^{t \cos at}, e^{t \sin at}) = \begin{vmatrix} e^{t \cos at} & e^{t \sin at} \\ -2e^{t \sin at} & 2e^{t \cos at} \end{vmatrix} = e^{2t} \left[ e^{t \cos at} \left( e^{t \cos at} + e^{t \sin at} \right) - e^{t \sin at} \left( -e^{t \cos at} + e^{t \sin at} \right) \right] \]

\[ = 2e^{2t} \cos at + e^{2t} \cos at \sin 2t + 2e^{2t} \sin at - e^{2t} \cos at \sin 2t \]

\[ = 2e^{2t} \left( \cos at + \sin 2t \right) = 2e^{2t} \neq 0 \]

\[ \Rightarrow \{ e^{t \cos at}, e^{t \sin at} \} \]

are linearly independent
#23. \( y'' - 2y' + 5y = 0 \)

\[
y_1 = e^t \cos 2t \\
y_1' = -2e^t \sin 2t + e^t \cos 2t \\
y_1'' = -4e^t \cos 2t - 2e^t \sin 2t + e^t \cos 2t - 2e^t \sin 2t \\
y_1''' = -8e^t \cos 2t - 4e^t \sin 2t - 2e^t \cos 2t - 2e^t \sin 2t \\
-2y_1'' = 2e^t \cos 2t - 4e^t \sin 2t + 5e^t \cos 2t \\
y_1'' - 2y_1' + 5y_1 = 0
\]

Also show this for \( y_2 = e^t \sin 2t \)

Since \( e^t \cos 2t \) and \( e^t \sin 2t \) are solutions to DE and are linearly independent and it is a 2nd order DE, they form a fundamental set of solutions.
#15. \( f_1(t) = 5, f_2(t) = \cos^2 t, f_3(t) = \sin^2 t \)

\[
\begin{vmatrix}
5 & \cos^2 t & \sin^2 t \\
0 & -2\cos t \sin t & 2 \sin t \cos t \\
0 & 2 \sin^2 t & -2 \sin^2 t + 2 \cos^2 t \\
\end{vmatrix} = 0 \Rightarrow \{5, \cos^2 t, \sin^2 t\} \text{ are linearly dependent}
\]

OR \[
5 - 5 \cos^2 t - 5 \sin^2 t = 5 - 5(\cos^2 t + \sin^2 t) = 5 - 5 = 0
\]

\[\Rightarrow \{5, \cos^2 t, \sin^2 t\} \text{ is linearly dependent}\]
f1 = \cos(2t); \ f2 = 1; \ f3 = (\cos \ t)^2

W(f1, f2, f3) =
4(2\sin(t)\cos(t)\cos(2t) - [2(\cos(t))]^2 - 1)\sin(2t) =

4[\sin(2t)\cos(2t) - \cos(2t)\sin(2t)] = 4[0] = 0
3.2 Homogeneous Linear Equations with Constant Coefficients

Method of Solution \[ \frac{dy}{dt} = -cy \] constant \[ \frac{dy}{dt} = ky \]

We know that \( \frac{dy}{dt} + ay = 0 \) has the solution \( y = c_1 e^{-at} \).

Do higher order linear equations also have exponential solutions?
Yes, all solutions are or are constructed out of exponential solutions.
1. Case 1: Distinct Real Roots

Assume \( ay'' + by' + cy = 0 \) has solution \( y = e^{mt} \). Then
\[
y' = me^{mt}; \quad y'' = m^2 e^{mt}.
\]

Then \( am^2 e^{mt} + bme^{mt} + ce^{mt} = 0 \) or \( e^{mt} (am^2 + bm + c) = 0 \).

\[ am^2 + bm + c = 0 \] is called the auxiliary equation or the characteristic equation.
Example 1: Solve the linear homogeneous DE

\[ 8y'' + 2y' - y = 0. \]

Assume \( y = e^{mt} \), \( y' = me^{mt} \), \( y'' = m^2 e^{mt} \);

Show:

Auxiliary Equation: \( 8m^2 + 2m - 1 = 0 \)

\[ 8(m^2 e^{mt}) + 2(me^{mt}) - e^{mt} = 0 \]

\[ e^{mt} [8m^2 + 2m - 1] = 0 \]

\[ 8m^2 + 2m - 1 = 0 \quad m = \frac{1}{4} \]

\[ (4m-1)(2m+1) = 0 \quad m = -\frac{1}{2} \]

General Solution:

\[ y = c_1 e^{\frac{1}{4}t} + c_2 e^{-\frac{1}{2}t} \]

\( e^{\frac{1}{4}t} \) and \( e^{-\frac{1}{2}t} \) are linearly independent.
Case 2: Repeated Real Roots \( y = c_1 e^{rt} + c_2 te^{rt} \)

Proof: If \( y_1, y_2 \) are linearly independent, \( y_i \) cannot be constant so that \( y_2' = m_i e^{rt} + u't e^{rt} \).

\[
y_2' = m_i e^{rt} + u't e^{rt}
\]

\[
y_2'' = m_i^2 e^{rt} + u''t e^{rt}
\]

Substitute in \( ay'' + by' + cy = 0 \) to obtain...

\[
e^{rt} [ (am_i + b)u' + (an_i^2 + bm_i + c)u ] = 0
\]

\[
m_i = \frac{-b}{2a}
\]

\[
2am_i + b = \frac{2a(-b)}{2a} + b = 0
\]

\[
2am_i + b = 0
\]

\[
au'' = 0 \iff u'' = 0 \text{ (integrate) } \Rightarrow u = c_2 t + c_1
\]

\[
u' = c_2 \text{ (integrate) } \Rightarrow u = c_1 + c_2 t
\]
\[ u = C_1 + C_2 t \; \text{, we know } y_1 = e^{mt} \text{ is a solution;} \]
\[ y_2 = uy_1 \]

\[ C_1 e^{mt} \text{ is already a solution; obtain second solution by letting } C_1 = 0, C_2 = 1. \]

\[ u = C_1 + C_2 t \]
\[ u = 0 + (1)t \Rightarrow u = t \]

\[ y_2 = te^{mt} \text{? linearly independent} \]

\[ y_1 = e^{mt} \]

**General solution:** \[ y = c_1 e^{mt} + c_2 te^{mt} \]
Example 2: Solve the linear homogeneous DE
\[
\frac{d^2 y}{dt^2} - 10 \frac{dy}{dt} + 25 y = 0
\]
\[y = e^{mt}\]

Auxiliary Equation:
\[
m^2 - 10m + 25 = 0
\]
\[
(m - 5)^2 = 0 \quad m = 5, 5 \quad \text{repeated}
\]

General Solution:
\[
y = c_1 e^{5t} + c_2 te^{5t}
\]
Case 3: Conjugate Complex Roots.

\[ y = \frac{1}{m_1 - m_2} \]
\[ m_1 = \alpha + \beta i \quad m_2 = \alpha - \beta i \]

I could write solution as:
\[ y = c_1 e^{(\alpha + \beta i)t} + c_2 e^{(\alpha - \beta i)t} \quad \text{OR} \]

Use Euler's Formula:
\[ e^{\theta} = \cos \theta + i \sin \theta \]

\[ e^{\beta t} = \cos (\beta t) + i \sin (\beta t) \]
\[ e^{-\beta t} = \cos (\beta t) - i \sin (\beta t) \]
\[ e^{\beta t} + e^{-\beta t} = 2 \cos \beta t \]
In \[ y = c_1 e^{\alpha t} + c_2 e^{-\beta t} \]

\[ = e^{\alpha t} [c_1 e^{-\beta t} + c_2 e^{\beta t}]\]

Let \( c_1 = c_2 = 1 \)

\[ \Rightarrow \text{one solution is } y_1 = e^{\alpha t} e^{\beta t} \]

\[ e^{\alpha t} - e^{-\beta t} = 2 \alpha e^{\beta t} \sin \beta t \]

Let \( c_1 = 1 ; c_2 = -1 \)

a second solution is \( y_2 = e^{\alpha t} e^{-\beta t} \)

General solution: \( y = e^{\alpha t} [c_1 \cos \beta t + c_2 \sin \beta t] \)
Example 3: Solve the linear homogeneous DE
\[2y'' - 3y' + 4y = 0.\]

Auxiliary Equation: \[2m^2 - 3m + 4 = 0\]

Use quadratic formula:
\[m = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(4)}}{2(2)} = \frac{3 \pm \sqrt{-23}}{4}\]

\[m = \frac{3}{4} \pm \frac{i\sqrt{23}}{4}\]

General solution:
\[y = e^{\frac{3}{4}t} [c_1 \cos \left(\frac{\sqrt{23}}{4} t\right) + c_2 \sin \left(\frac{\sqrt{23}}{4} t\right)]\]
Example 4: Solve the linear homogeneous DE
\[ y'' + y = 0. \]

Auxiliary Equation: 
\[ m^2 + 1 = 0 \]
\[ m = \pm i \]
\[ = 0 \pm 1i \]

General solution: 
\[ y = e^{ix} [c_1 \cos t + c_2 \sin t] \]
\[ y = c_1 \cos t + c_2 \sin t \]
Higher Order Equations

An $n$th degree linear homogeneous DE has an $n$th degree auxiliary equation. Solutions will involve terms of the form $e^{mt}, te^{mt}, \ldots, t^{n-1} e^{mt}, \cos bt, \sin bt, t \cos bt, t \sin bt, \ldots$

(and their products)
Examples: Solve the linear homogeneous DE's

5. \[ \frac{d^5 y}{dt^5} - 16 \frac{dy}{dt} = 0 \]

Auxiliary Equation: \[ m^5 - 16m = 0 \]
\[ m(m^4 - 16) = m(m^2 - 4)(m^2 + 4) = 0 \]
\[ m = 0, \pm 2 \pm 2i \]
\[ \alpha = 0 \]
\[ \beta = 2 \]

General solution: \[ y = c_1 e^{-2t} + c_2 e^{2t} - 2t + c_3 e^{-2t} + c_4 \cos 2t + c_5 \sin 2t \]

\[ y = c_1 + c_2 e^{2t} + c_3 e^{-2t} + c_4 \cos 2t + c_5 \sin 2t \]
6. \( y'' + 2y' - 5y - 6y = 0 \) with \( y(0) = y'(0) = 0; y''(0) = 1. \)

**Auxiliary Equation:**

\[ m^3 + 2m^2 - 5m - 6 = 0 \]
\[ m = 2, -1, -3 \]

General solution:

\[ y = c_1 e^{-3t} + c_2 e^{-t} + c_3 e^{2t} \]

\[ y = c_1 e^{-3t} + c_2 e^{-t} + c_3 e^{2t} \]
\[ y(0) = 0 = c_1 + c_2 + c_3 \]
\[ y' = -3c_1 e^{-3t} - c_2 e^{-t} + 2c_3 e^{2t} \]
\[ y'(0) = 0 = -3c_1 - c_2 + 2c_3 \]
\[ y'' = 9c_1 e^{-3t} - c_2 e^{-t} + 4c_3 e^{2t} \]
\[ y''(0) = 1 = 9c_1 + c_2 + 4c_3 \]

System of 3 equations with 3 unknowns.
\[ c_1 + c_2 + c_3 = 0 \]
\[ -3c_1 - c_2 + 2c_3 = 0 \]
\[ 9c_1 + c_2 + 4c_3 = 1 \]

Augmented matrix:
\[
\begin{bmatrix}
1 & 1 & 0 \\
-3 & -1 & 2 & 0 \\
9 & 1 & 4 & 1
\end{bmatrix}
\]
3x4

(Reduced Row Echelon Form)
\[
\text{ref}
\]

\[ c_1 = \frac{1}{10} \]
\[ c_2 = -\frac{1}{6} \]
\[ c_3 = \frac{1}{15} \]

Solution to IVP:
\[
y = \frac{1}{10} e^{-3t} - \frac{1}{6} e^{-t} + \frac{1}{15} e^{2t}
\]

Homework (Section 3.2)