Section 1.1 (Homework)

#17. \[ y' = y(1-y) \]

\[ y = \frac{1}{1+c_1e^{-t}} = (1+c_1e^{-t})^{-1} \]
\[ y' = -1 \frac{c_1e^{-t}}{(1+c_1e^{-t})^2} (-c_1e^{-t}) \]
\[ y' = \frac{c_1e^{-t}}{(1+c_1e^{-t})^2} \]

\[ y(1-y) = \frac{1}{1+c_1e^{-t}} \left( 1 - \frac{1}{1+c_1e^{-t}} \right) \]
\[ = \frac{1}{1+c_1e^{-t}} \left( \frac{1+c_1e^{-t}-1}{1+c_1e^{-t}} \right) \]
\[ = \frac{c_1e^{-t}}{(1+c_1e^{-t})^2} \]

\[ \checkmark \]
#13. $y'' - 6y' + 13y = 0$; $y = e^{3t} \cos 2t$

$y' = e^{3t}(-2\sin 2t) + 3e^{3t} \cos 2t$

$y'' = -2e^{3t}(2\cos 2t) - 6e^{3t} \sin 2t$

$+ 3e^{3t}(-2\sin 2t) + 9e^{3t} \cos 2t$

$y'' = 5e^{3t} \cos 2t - 12e^{3t} \sin 2t$

$- 6y' = -18e^{3t} \cos 2t + 12e^{3t} \sin 2t$

$+ 13y = 13e^{3t} \cos 2t$

$y'' - 6y' + 13y = 0 \checkmark$
Problem 21

\[
\begin{align*}
\frac{dx}{dt} &= x + 3y \
\frac{dy}{dt} &= 5x + 3y \\
\text{system} \\
\frac{dx}{dt} &= -2e^{-2t} + 18e^{6t} \\
x + 3y &= (e^{-2t} + 3e^{6t}) + 3(-e^{-2t} + 5e^{6t}) \\
&= -2e^{-2t} + 18e^{6t} = \frac{dx}{dt} \quad \checkmark \\
\frac{dy}{dt} &= 2e^{-2t} + 30e^{6t} \\
5x + 3y &= 5(e^{-2t} + 3e^{6t}) + 3(-e^{-2t} + 5e^{6t}) \\
&= 2e^{-2t} + 30e^{6t} = \frac{dy}{dt} \quad \checkmark \\
\end{align*}
\]

We have verified that the given \((x(t), y(t))\) is a solution to the given system of DE's.
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1.2 Initial-Value Problems

An nth order initial value problem (IVP) may be phrased as follows:

On some interval $I$ containing $t_0$, solve the DE

\[
\frac{d^n y}{dx^n} = f(t, y, y', \ldots, y^{(n-1)}) \quad \text{subject to}
\]

\[
y(t_0) = y_0, \ y'(t_0) = y_1, \ldots, y^{(n-1)}(t_0) = y_{n-1} \quad \text{where}
\]

$y_0, y_1, \ldots, y_{n-1}$ are arbitrarily chosen real constants.

Example: Use the fact that $x = c_1 \cos t + c_2 \sin t$ is a two-parameter family of solutions of the DE $x'' + x = 0$ to find a solution with initial conditions $x(\frac{\pi}{2}) = 0, x'(\frac{\pi}{2}) = 1$.

\[
x(\frac{\pi}{2}) = c_1 \cos \frac{\pi}{2} + c_2 \sin \frac{\pi}{2} = 0
\]

\[
= c_1 (0) + c_2 (1) = 0 \quad \Rightarrow c_2 = 0
\]

\[x = c_1 \cos t \]

\[x' = -c_1 \sin t \]

\[x'(\frac{\pi}{2}) = -c_1 \sin \frac{\pi}{2} = -c_1 (1) = 1 \]

\[\Rightarrow c_1 = -1 \]

particular solution to this IVP is $x = -\cos t$
Existence and Uniqueness:

Existence: Does the DE \( \frac{dy}{dx} = f(t, y) \) possess solutions? Do any of the solutions pass through the point \((t_0, y_0)\)?

Uniqueness: When can we be certain that there is precisely one solution curve passing through the point \((t_0, y_0)\)?
Theorem 1.1. Existence of a Unique Solution

Let $R$ be a rectangular region in the $ty$-plane defined by $a \leq t \leq b; c \leq y \leq d$ that contains the point $(t_0, y_0)$ in its interior. If $f(t, y)$ and $\frac{\delta f}{\delta y}$ are continuous on $R$, then there exists some interval $I_0 : t_0 - h < t < t_0 + h, h > 0$ contained in $a \leq t \leq b$ and a unique function $y(t)$ defined on $I_0$ that is a solution of the initial value problem.

(See Examples 3 and 4 on pages 14 – 15.)
Intervals of Existence and Uniqueness:

The following 3 sets on the real $t$-axis may not be the same:

1. The domain of the function $f(t)$.
2. The interval $I$ over which the solution $y(t)$ exists or is defined.
3. The interval $I_0$ of existence and uniqueness.

The interval of existence $I$ is the largest interval containing $t_0$ over which the solution $y(t)$ is defined and differentiable. The solution $y(t)$ is unique in the local sense; that is, a solution near the point $(t_0, y_0)$. 
Examples: Determine a region of the $ty$-plane for which the given DE will have a unique solution whose graph passes through a point $(t_0, y_0)$ in the region.

\[
\frac{dy}{dt} = y 
\]

\[
\frac{dy}{dt} = \frac{y}{t} = f(t, y) \quad t \neq 0
\]

1. \[
\frac{\delta f}{\delta y} = \frac{1}{t} \quad t \neq 0
\]

are which is continuous for $t \neq 0$.

The DE will have a unique solution in either of the half-planes $t > 0$ or $t < 0$. 
2. \((t^2 + y^2)y' = y^2\)

\[
y' = \frac{y^2}{t^2 + y^2} = f(t, y)
\]

continuous except at \((0, 0)\)

\[
\frac{\partial f}{\partial y} = \frac{(t^2 + y^2)(2y) - y^2(2y)}{(t^2 + y^2)^2}
\]

\[
= \frac{2t}{t^2 + y^2}
\]

continuous except at \((0, 0)\)

\((t^2 + y^2)y' = y^2\) has a unique solution in any region that does not contain \((0, 0)\)
1. \( t \frac{dy}{dt} = y \)
\( \frac{dy}{y} = \frac{dt}{t} \)

4. \( \int \frac{y}{t} \, dt \)

B. Solve \( ty' = y \)

If \( y = c_1 e^t \)
\( y(0) = c_1 \)

\( t(c_1) = c_1 t \)
2. \((t^2 + y^2)y' = y^2\)

\[ y' = \frac{y}{t^2 + y^2} = f(t, y) \]

\[ \frac{df}{dy} = \frac{\frac{y^2}{t^2 + y^2}}{dy} \]

4. Solve: \((t^2 + y^2)y' = y^2\)

\[ \text{implicit solution} \]

you may try \(F2\) solve to find an explicit solution.

Homework Section 1.2
2.1 What a First-Order Derivative Can Tell Us

Direction Fields (Slope Fields)

Example: \( \frac{dy}{dt} = 3y \) has the form \( f(t,y) = c = 3y \)

If \( c = 3, y = 1 \). This says that along the horizontal line \( y = 1 \), the slopes of the tangents to the solution curve are all 3.

If \( c = 1, y = 1/3 \). Along the horizontal line \( y = 1/3 \), the slopes of the tangents to the solution curve are all 1.
The small slope line segments are called lineal elements. The collection of all lineal elements for a given DE is called a direction field or slope field.
You may use your calculator to graph the direction field. The TI calculators use *Slope Field* for single DE’s and *Direction Field* for systems of DE’s.
Example:
\[ y' = y \]
\[ y(0) = 2 \]

Calculator
\[ t_0 = 0 \]
\[ y_1' = y_1 \]
\[ y_1 = 0 \]
Example: \( \frac{dy}{dt} = 3y \)

Calculator: \( y' = 3y \)

Direction Field
Example: Use a graphing utility to find the direction field for the DE $y' = t + y$. Sketch the solution that passes through $(0, 1)$. 
Phase Portraits and Stability

Let \( t = \) independent variable
\( x = \) dependent variable

A first-order DE of the form \( F(x, x') = 0 \) is called autonomous (the independent variable does not appear explicitly). If this can be solved for \( x' = \frac{dx}{dt} \), we can write

\[
\frac{dx}{dt} = g(x) \quad \text{and the solution is} \quad x(t), \quad \text{where both} \quad g \quad \text{and} \quad g' \\
\text{are assumed to be continuous functions of} \quad x \quad \text{on some interval} \quad I \quad \text{implying that there is a unique solution through} \quad (t_0, x_0).
\]
We say that $x_1$ is a **critical point** (or **equilibrium point** or **stationary point**) of the DE if $g(x_1) = 0$.

If $x_1$ is a critical point, $x(t) = x_1$ is a constant solution (or **equilibrium solution**) of the autonomous

equation: $\frac{dx}{dt} = g(x_1) = 0$. (These are the only constant solutions to the DE.)
For an autonomous DE, we have:

1. Since the DE is independent of $t$, at all points on any line parallel to the $t$-axis, the slopes are all the same. Thus, if $x(t)$ is a solution, so is $x(t-a)$.

2. Since $g$ and $g'$ are continuous, Theorem 1.1 holds in some horizontal strip or region in the $tx$-plane. Thus, through any point $(t_0, x_0)$ in this region, there passes only one solution curve.

\[ x = x, \quad \text{(Constant solution)} \]
Example 2 (from the text): \[
\frac{dx}{dt} = x(a - bx)
\]

Example 2 (from the text):

\[a, b > 0\]

The critical points are 0 and \(a/b\).

\[
\begin{array}{c|c|c|c}
\text{Interval} & \text{Sign} & x(t) & \text{arrow} \\
\hline
\hline
a/b & & & \\
0 & & & \\
\end{array}
\]

Phase portrait

\[
\text{Set } 0 \\
X = 0 \\
a - bx = 0 \\
X = \frac{a}{b} \\
\]

Phase portrait
\[ \frac{dx}{dt} = x(3 - x) \]