Section 9.5 (Homework)

\[ \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} = \frac{\pi}{4} \]

\[
\begin{array}{c|c}
\text{n} & \text{Sn} \\
\hline
1 & 1 \\
2 & .6667 \\
3 & .8667 \\
4 & .7238 \\
5 & .8349 \\
6 & \text{etc.}
\end{array}
\]
distances between $S_n$ and the line $y = \frac{\pi}{4}$ decrease as $n$ increases.

Example: $S_9 = 0.81309 \approx 0.8131$ in $L_2$ for $n=10$, $|a_{10}| = 0.0526$

Thm 9.15: $|S_n - \frac{\pi}{4}| < |a_{10}|$

erate, $|0.8131 - \frac{\pi}{4}| = 0.0277 < 0.0526$
#29. \( \sum_{n=1}^{\infty} \frac{(-1)^n n!}{1 \cdot 3 \cdot 5 \cdots (2n-1)} \)

\[
\frac{|a_{n+1}|}{a_n} = \frac{(n+1)!}{1 \cdot 3 \cdot 5 \cdots (2n+1)} \cdot \frac{h!}{h!} = \frac{(n+1)!}{1 \cdot 3 \cdot 5 \cdots (2n-1) (2n+1)} + \frac{3 \cdot 5 \cdots (2n-1)}{h!}.
\]

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{(n+1)!}{h!} \cdot \frac{1}{n} = \frac{1}{2} < 1 \Rightarrow \text{series converges by Ratio Test.}
\]
Section 9.6 (continued)

Theorem 9.17  Ratio Test

Let \( \sum a_n \) be a series with nonzero terms.

1. \( \sum a_n \) converges absolutely if \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1 \).

2. \( \sum a_n \) diverges if \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| > 1 \) or \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty \).

3. The Ratio Test is inconclusive if \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1 \).
Use the Ratio Test to determine if the series converges or diverges.

Examples:

\[
\sum_{n=0}^{\infty} \frac{(-3)^n}{n!} \quad |a_n| = \frac{3^n}{n!}.
\]

\[
|a_{n+1}| = \frac{3^{n+1}}{(n+1)!}.
\]

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{3^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n} = \lim_{n \to \infty} \frac{3}{n+1} = 0 < 1
\]

By the Ratio Test, the series converges.
\[ \sum_{n=1}^{\infty} \frac{n^3}{2^n} \]

\[ |a_n| = \frac{n^3}{2^n} \]

\[ |a_{n+1}| = \frac{(n+1)^3}{2^{n+1}} \]

\[ \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{(n+1)^3}{2^{n+1}} \cdot \frac{2^n}{n^3} = \frac{2^n}{n^3} \]

\[ = \lim_{n \to \infty} \frac{1}{2} \left[ \frac{n^3 + 3n^2 + 3n + 1}{n^3} \right] \]

\[ = \lim_{n \to \infty} \frac{1}{2} \left[ 1 + \frac{3}{n} + \frac{3}{n^2} + \frac{1}{n^3} \right] = \frac{1}{2} < 1 \]

The series converges by the Ratio Test.
\[ \sum_{n=1}^{\infty} \frac{n^n}{n!} \]

\[ |a_n| = \frac{n^n}{n!} \quad \text{and} \quad |a_{n+1}| = \frac{(n+1)^{n+1}}{(n+1)!} \]

(Find \( y = \lim_{n \to \infty} \frac{a_{n+1}}{a_n} \) and then take \( \ln \) of both sides.)

\[ \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{h^n} \]

\[ = \lim_{n \to \infty} \frac{(n+1)^{n+1}}{(n+1)^{n+1}} \cdot \frac{n!}{h^n} \]

\[ = \lim_{n \to \infty} \left( \frac{n+1}{n} \right)^n = \lim_{n \to \infty} \left( 1 + \frac{1}{n} \right)^n \]

Let \( y = \lim_{n \to \infty} \left( 1 + \frac{1}{n} \right)^n \)
\[\begin{align*}
\text{Let } y &= \lim_{n \to \infty} (1 + \frac{1}{n})^n \\
\ln y &= \ln \lim_{n \to \infty} (1 + \frac{1}{n})^n \\
&= \lim_{n \to \infty} \ln (1 + \frac{1}{n}) \\
&= \lim_{n \to \infty} n \ln (1 + \frac{1}{n}) \\
&= \lim_{n \to \infty} \frac{\ln (1 + \frac{1}{n})}{\frac{1}{n}} \\
&= \lim_{n \to \infty} \frac{\ln (1 + \frac{1}{n})}{\frac{1}{n}} (\frac{0}{0})
\end{align*}\]
we have, \( \ln y = \ln \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1 \)

\[ y = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = e' = e > 1 \]

\[ \Rightarrow \text{By the Ratio Test, } \sum_{n=1}^{\infty} \frac{n!}{n^n} \text{ diverges} \]

\[ \Rightarrow \sum_{n=1}^{\infty} \frac{n!}{h^n} \text{ converges} \]

Homework Section 9.6
9.8 Power Series

Definition of Power Series:

If \( x \) is a variable, then an infinite series of the form

\[
\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots
\]

is called a power series. More generally, series of the form

\[
\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1 (x-c) + a_2 (x-c)^2 + \ldots + a_n (x-c)^n + \ldots
\]

is called a power series centered at \( c \), where \( c \) is a constant.
Theorem 9.20  Convergence of a Power Series

For a power series centered at $c$, precisely one of the following is true.

1. The series converges only at $c$.
2. There exists a real number $R > 0$ such that the series converges absolutely for $|x - c| < R$, and diverges for $|x - c| > R$.
3. The series converges absolutely for all $x$.

The number $R$ is called the radius of convergence of the power series. If the series converges only at $c$, the radius of convergence $R = 0$, and if the series converges for all $x$, the radius of convergence is $R = \infty$. The set of all values of $x$ for which the series converges is the interval of convergence of the power series.
Examples: Find the radius of convergence and the interval of convergence. (Start with Ratio Test)
\[ |a_n| = \frac{|x|^n}{n^2} \quad |a_{n+1}| = \frac{|x|^{n+1}}{(n+1)^2} \]
\[
\sum_{n=1}^{\infty} \frac{x^n}{n^2}
\]
\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{|x|^{n+1}}{(n+1)^2} \cdot \frac{n^2}{|x|^n} \cdot \frac{1}{n^2} \frac{1}{(n^2 + 2an + 1)} \text{ use Ratio Test}
\]
\[
= |x| \lim_{n \to \infty} \frac{1}{1 + \frac{2a}{n} + \frac{1}{n^2}} = |x| \cdot 1 < 1
\]
\[
\Leftrightarrow |x-0| < 1 \quad \text{or} \quad |x| < 1
\]
\[ R = 1 \]
At this point, we know:

1. $R = 1$
2. Series converges for $|x| < 1$
   - $-1 < x < 1$
   - Series diverges for $|x| > 1$

We need to check the endpoints: $x = \pm 1$

1. $x = 1$
   \[
   \sum_{n=1}^{\infty} \frac{x^n}{h^2} = \sum_{n=1}^{\infty} \frac{1^n}{h^2} = \frac{1}{h^2}
   \]
   - p-series, $p = 2$

2. $x = -1$
   \[
   \sum_{n=1}^{\infty} \frac{(-1)^n}{h^2}
   \]
   - Converges absolutely
   \[
   = \text{it converges}
   \]

Interval of convergence: $[-1, 1]$
\[
\sum_{n=0}^{\infty} n^3 (x-5)^n
\]

\[
\frac{|a_{n+1}|}{|a_n|} = \lim_{n \to \infty} \frac{(n+1)^3 |x-5|^{n+1}}{n^3 |x-5|^n}
\]

\[
= |x-5| \lim_{n \to \infty} \left( \frac{(n+1)^3}{n^3} \right)
\]

Apply Ratio Test

\[
= |x-5| \lim_{n \to \infty} \left[ 1 + \frac{3}{n} + \frac{3}{n^2} + \frac{1}{n^3} \right] = |x-5| \cdot 1
\]

Series converges for \( |x-5| < 1 \)
\[
\implies -1 < x-5 < 1
\]
\[
\implies 4 < x < 6
\]

Series diverges for \( |x-5| > 1 \)
Check endpoints: $x = 4, x = 6$

$$\sum_{n=1}^{\infty} n^3 (x - 5)^n$$

Let $x = 6$  
$$\sum_{n=1}^{\infty} n^3 \cdot 1 = \sum_{n=1}^{\infty} n^3$$

$$\lim_{n \to \infty} n^3 = \infty$$, series diverges

by $n$th term test

Let $x = 4$  
$$\sum_{n=1}^{\infty} n^3 (-1)^n$$

$$\lim_{n \to \infty} n^3 (-1)^n \neq 0$$

series diverges

$R = 1$

interval of convergence: $(4, 6)$
\[ \sum_{n=1}^{\infty} \frac{(x-4)^n}{n5^n} \]

\[ |a_n| = \frac{|x-4|^n}{n5^n} \]

\[ |a_{n+1}| = \frac{|x-4|^{n+1}}{(n+1)5^{n+1}} \]

\[ \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{|x-4|^n}{(n+1)5^n} \cdot \frac{5}{|x-4|^n} \]

\[ = \frac{|x-4|}{5} \lim_{n \to \infty} \left| \frac{(n+1)5^n}{n5^n} \right| \frac{1}{h(n)} \]

\[ = \frac{|x-4|}{5} \lim_{n \to \infty} \frac{1}{1+\frac{1}{h}} = \frac{|x-4|}{5} \cdot 1 < 1 \]

\[ \Rightarrow \quad |x-4| < 5 \]

From \[ |x-c| < R \]

\[ \text{series converges for } |x-4| < 5 \]

\[ -5 < x-4 < 5 \]

\[ -1 < x < 9 \]
\[ \sum_{n=1}^{\infty} \frac{(x-4)^n}{n5^n} \]

Check endpoints:
\[ x = -1, 9 \]

\[ x = 9 \]
\[ \sum_{n=1}^{\infty} \frac{(9-4)^n}{n5^n} = \sum_{n=1}^{\infty} \frac{5^n}{n5^n} = \sum_{n=1}^{\infty} \frac{1}{n} = \infty \]

Divergent harmonic series \((p=1)\)

\[ x = -1 \]
\[ \sum_{n=1}^{\infty} \frac{(-1-4)^n}{n5^n} = \sum_{n=1}^{\infty} \frac{(-5)^n}{n5^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n5^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} = \frac{\pi}{2} \]

Convergent alternating harmonic series

\[ R = 5 \]
Interval of convergence \([-1, 9)\)