Section 5.2 (Homework)

#25. \[ \int \frac{1}{1 + \sqrt{2x}} \, dx \]

Let \( u = 1 + \sqrt{2x} \)
\[ u - 1 = \sqrt{2x} \]
\[ (u-1)^2 = (\sqrt{2x})^2 \]
\[ (u-1)^2 = 2x \]
\[ \frac{1}{2} (u-1) = x \]
\[ \frac{1}{2} \cdot 2(u-1) \, du = dx \]

\[ \int \frac{1}{1 + \sqrt{2x}} \, dx = \int \frac{1}{u} (u-1) \, du \]

\[ = \int [1 - \frac{1}{u}] \, du = u - \ln |u| + C \]
\[ = (1 + \sqrt{2x}) - \ln (1 + \sqrt{2x}) + C \]
\( (C = 1 + C_1) \)

\[ = \sqrt{2x} - \ln (1 + \sqrt{2x}) + C \]
5.3 Inverse Functions

Definition: A function $g$ is the inverse function of the function $f$ if $f(g(x)) = x$ for each $x$ in the domain of $g$ and $g(f(x)) = x$ for each $x$ in the domain of $f$. The function $g$ is denoted by $f^{-1}$ (read “$f$ inverse”).

- Domain of $f = \text{Range of } g = f^{-1}$
- Range of $f = \text{Domain of } g = f^{-1}$
The graph of \( f \) contains the point \((a, b)\) if and only if the graph of \( f^{-1} \) contains the point \((b, a)\).

(The graph of \( f^{-1} \) is a reflection of the graph of \( f \) in the line \( y = x \).)

\[
\text{ex. } \frac{f(x)}{f(f^{-1}(x))} = x + 2 \Rightarrow f^{-1}(x) + 2 = f^{-1}(x) + 2 = x - 2 + 2 = x
\]
The Existence of an Inverse Function

1. A function has an inverse function if and only if it is one-to-one. (For each \( y \) there is exactly one \( x \))

2. If \( f \) is strictly monotonic on its entire domain, then it is one-to-one and therefore has an inverse function.

monotonic: either strictly increasing or strictly decreasing (usually)

\[
\begin{align*}
\text{exception:} & \quad \text{increasing:} \\
& \quad \text{if } x_1 < x_2, \\
& \quad \text{then } f(x_1) \leq f(x_2)
\end{align*}
\]
**Horizontal Line Test** A function is one-to-one if no horizontal line intersects the graph more than once.

- Both are functions (vertical line test)
- Increasing $\Rightarrow$ monotonic $\Rightarrow$ one-to-one
- Not monotonic $\Rightarrow$ not one-to-one
Examples:

Show analytically and graphically that $f$ and $g$ are inverses if:

$$f(x) = 16 - x^2, \quad x \geq 0 \quad \text{and} \quad g(x) = \sqrt{16 - x}$$

$$f(g(x)) = 16 - (g(x))^2 = 16 - (\sqrt{16 - x})^2$$
$$= 16 - 16 + x = x \quad \checkmark$$

$$g(f(x)) = \sqrt{16 - f(x)} = \sqrt{16 - (16 - x^2)}$$
$$= \sqrt{x^2} = x \quad \checkmark$$

![Graph showing $y = x$](image)
Use the derivative to determine whether the function is strictly monotonic on its entire domain

if \( f(x) = x^3 - 6x^2 + 12x \)

\[
f'(x) = 3x^2 - 12x + 12
= 3(x^2 - 4x + 4)
= 3(x-2)^2 \geq 0 \quad x = 2
\]

otherwise \( f'(x) = 3(x-2)^2 > 0 \)

\[
\Rightarrow f \text{ is increasing}
\Rightarrow \text{ monotonic}
\Rightarrow \text{ one-to-one}
\]
Find the inverse function and graph both \( f \) and its inverse if

\[
f(x) = 3^{\frac{5}{2}}x - 1
\]

1. Let \( y = f(x) = 3^{\frac{5}{2}}x - 1 \)

2. Interchange \( x \) and \( y \):
   \[
x = 3^{\frac{5}{2}}y - 1
   \]

3. Solve for \( y \):
   \[
   \frac{x}{3} = (3^{\frac{5}{2}}y - 1)^{\frac{1}{5}}
   \]
   \[
   \frac{x}{3} = \sqrt[5]{2y - 1}
   \]
   \[
   \frac{x^5}{243} = 2y - 1
   \]
   \[
   \frac{x^5}{243} + 1 = 2y
   \]
   \[
   \frac{1}{2}(\frac{x^5}{243} + 1) = y
   \]

4. The new "\( y \)" is \( f^{-1}(x) \):
   \[
   f^{-1}(x) = \frac{x^5}{486} + \frac{1}{2}
   \]
\[ f(x) = 3 \sqrt[5]{2x-1} \]
\[ f'(x) = \frac{x^5}{486} + \frac{1}{2} \]

**Check:**
\[ f(f'(x)) = 3 \sqrt[5]{2f'(x) - 1} \]
\[ = 3 \sqrt[5]{2 \left( \frac{x^5}{486} + \frac{1}{2} \right) - 1} \]
\[ = 3 \sqrt[5]{\frac{x^5}{243} + 1 - 1} \]
\[ = 3 \sqrt[5]{\frac{x^5}{243}} = 3 \cdot \frac{x}{3} = x \]
Theorem 5.8  Continuity and Differentiability of Inverse Functions

1. If \( f \) is continuous on its domain, then \( f^{-1} \) is continuous on its domain.
2. If \( f \) is increasing on its domain, then \( f^{-1} \) is increasing on its domain.
3. If \( f \) is decreasing on its domain, then \( f^{-1} \) is decreasing on its domain.
4. If \( f \) is differentiable at \( c \), then \( f^{-1} \) is differentiable at \( f(c) \).
Theorem 5.9  The Derivative of an Inverse Function

Let $f$ be a function that is differentiable on an interval I. If $f$ has an inverse function $g$, then $g$ is differentiable at any $x$ for which $f'(g(x)) \neq 0$.

$$g'(x) = \frac{1}{f'(g(x))}, f'(g(x)) \neq 0$$

\[ \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} \]
Example: \( f(x) = 2x + 3 \), \( a = 1 \)

Find \( (f^{-1})'(a) \)

If \( f^{-1}(i) = c \)

\[
\begin{align*}
1 &= f(c) = 2c + 3 \\
-2 &= 2c \quad \Rightarrow -1 = c
\end{align*}
\]

\( \Rightarrow \ f(-1) = 1 \) and \( f^{-1}(1) = -1 \)

Thm. 5.9 \( (f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))} \)

\[
\begin{align*}
f(x) &= 2x + 3 \\
f'(x) &= 2 = f'(-1) \\
\end{align*}
\]
OR: \( f(x) = 2x + 3 \)

\[
y = 2x + 3 \quad \text{(interchange x and y)}
\]

\[
x = 2y + 3
\]

\[
x - 3 = 2y \quad \text{or} \quad \frac{x - 3}{2} = y
\]

\[
f^{-1}(x) = \frac{x - 3}{2}
\]

\[
(f^{-1})'(x) = \frac{1}{2}
\]
Find $\left( f^{-1} \right)'(a)$ for $f(x) = \frac{1}{27}(x^5 + 2x^3)$ and $a = 11$. 

$-11 = \frac{1}{27}(x^5 + 2x^3)$

$-297 = x^5 + 2x^3$

$x^5 + 2x^3 + 297 = 0$

$\Rightarrow c = -3$

$f(x) = \frac{1}{27}(x^5 + 2x^3)$

$f'(x) = \frac{1}{27}(5x^4 + 6x^2) \geq 0 \Rightarrow f \uparrow$

(only one place where $f(x) = -11$)

$f(-3) = -11$

$-3 = f^{-1}(-11)$

Thm. 5.9:

$(f^{-1})'(-11) = \frac{1}{f'(-3)}$

$f'(-3) = \frac{1}{27}(5 \cdot 3^4 + 6 \cdot 3^2) = 17$

$f^{-1}(-3) = \frac{1}{27}(5 \cdot 3^4 + 6 \cdot 3^2) = 17$

$(f^{-1})'(-11) = \frac{1}{17}$
5.4 Exponential Functions: Differentiation and Integration

Definition: The inverse function of the natural logarithmic function $f(x) = \ln x$ is called the natural exponential function and is denoted by $f^{-1}(x) = e^x$.

That is, $y = e^x$ if and only if $x = \ln y$.

$\int (f(x)) = x$ ; $\int (f(x)) = x$

Note: $\ln(e^x) = x$; $e^{\ln x} = x$. 
Examples:

Solve for $x$ accurate to three decimal places:

$$e^{\ln(2x)} = 12$$

$$2x = 12 \implies x = 6$$

$$\ln(x^2) = 10$$

$$2\ln(x) = 10 \iff \ln x = 5$$

$$\iff e^{\ln x} = e^5$$

$$\implies x = e^5 \approx 148.413$$
Theorem 5.10 Operations with Exponential Functions

Let $a$ and $b$ be any real numbers:

1. $e^a e^b = e^{a+b}$
2. $\frac{e^a}{e^b} = e^{a-b}$

* Proof: Let $x = e^a$, $y = e^b$

$\Rightarrow \ln x = a$, $\ln y = b$

$\ln e^{a+b} = a+b = \ln x + \ln y$

$= \ln xy = \ln e^{a+b}$

$e^{a+b} = e^a e^b$

\[ e^{a+b} - e^a e^b \]
Properties of the Natural Exponential Function:

1. The domain of \( f(x) = e^x \) is \((-\infty, \infty)\), and the range is \((0, \infty)\).

2. The function \( f(x) = e^x \) is continuous, increasing, and one-to-one on its entire domain.

3. The graph of \( f(x) = e^x \) is concave upward on its entire domain.

4. \( \lim_{x \to -\infty} e^x = 0; \lim_{x \to \infty} e^x = \infty. \)

\[ \lim_{x \to \infty} \ln x = -\infty \]

\[ \lim_{x \to 0^+} \ln x \]
Theorem 5.11 The Derivative of the Natural Exponential Function

Let \( u \) be a differentiable function of \( x \).

1. \( \frac{d[e^x]}{dx} = e^x \)

2. \( \frac{d[e^u]}{dx} = e^u \frac{du}{dx} \)

Proof: Use implicit differentiation.

\[ y = e^x \iff \ln y = x \]

\[ \frac{1}{y} \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = y = e^x \]

\[ \frac{d[e^x]}{dx} = e^x \]
Examples: Differentiate.

\[ g(x) = e^{-5x} \cos 3x \]
\[ g'(x) = e^{-5x} [-\sin(3x) \cdot 3] \]
\[ + (-5)e^{-5x} \cos 3x \]
\[ - e^{-5x} [3 \sin 3x + 5 \cos 3x] \]

\[ h(\theta) = e^{\sin 5\theta} \]
\[ h'(\theta) = e^{\sin 5\theta} \cdot (\cos 5\theta \cdot 5) \]
\[ = 5e^{\sin 5\theta} \cos 5\theta \]
\[ y = \frac{e^x + e^{-x}}{e^x - e^{-x}} - 1 \]

\[
\frac{dy}{dx} = \frac{(e^{-x} - e^x)(e^x - e^{-x}) - (e^x + e^{-x})(e^x - e^{-x})}{(e^x - e^{-x})^2} 
\]

\[
= \frac{e^{-2x} - 2e^x + e^{2x} - 2e^{-x}}{(e^x - e^{-x})^2} 
\]

\[
= -4 \frac{2}{(e^x - e^{-x})^2} = -\left(\frac{2}{e^x - e^{-x}}\right)^2 
\]

\[
\text{Later} \quad \frac{d}{dx} \left[ \coth x \right] = -\csc^2 x 
\]
Theorem 5.12 Integration Rules for Exponential Functions

Let $u$ be a differentiable function of $x$.

1. $\int e^x \, dx = e^x + C$

2. $\int e^u \, du = e^u + C$
Integrate:

\[ \int_{0}^{1} xe^{-x^2} \, dx \]

Let \( u = -x^2 \)
\[ du = -2x \, dx \]
\[ u(0) = 0 \]
\[ u(1) = -1 \]
\[ \Rightarrow -1 = -1 \]

\[ = -\frac{1}{2} \left[ e^{-x^2} \right]_{0}^{1} \]
\[ = -\frac{1}{2} \left[ e^{-1} - e^{0} \right] \]
\[ \ln 1 = 0 \]
\[ 1 = e^{0} \]
\[ = -\frac{1}{2} \left[ e^{-1} - 1 \right] = \frac{-1}{2e} + \frac{1}{2} \]
\[
\int e^x \sin(e^x) \, dx \\
= \int \sin(e^x) (e^x \, dx) \\
= \int \sin(u) \, du = -\cos(u) + C \\
= -\cos(e^x) + C
\]

Let \( u = e^x \)
\( du = e^x \, dx \)

**Homework:** sections 5.3 and 5.4