Section 7.3 (Homework)

#17. $y = x^3$, $x = 0$, $y = 8$

Use shells.

$V = 2\pi \int_{0}^{8} y \cdot y' \, dy$

$= 2\pi \int_{0}^{8} y \, dy$

$= 2\pi \left[ \frac{y^2}{2} \right]_{0}^{8}$

$= 2\pi \left[ 64 - 0 \right]$

$= 128\pi$

$= \frac{128\pi}{7}$
Section 7.4 (Homework)

#11. \( y = \frac{1}{2} \left( e^x + e^{-x} \right) \) \([-1, 2]\)

\[
S = \int_a^b \sqrt{1 + \left[ f'(x) \right]^2} \, dx
\]

\[
y' = \frac{1}{2} \left( e^x - e^{-x} \right) \quad \text{and} \quad (y')^2 = \frac{1}{4} \left( e^{2x} - 2 + e^{-2x} \right)
\]

\[
 = \frac{1}{4} e^{2x} - \frac{1}{2} + \frac{1}{4} e^{-2x}
\]

\[
1 + (y')^2 = 1 + \frac{1}{4} e^{2x} - \frac{1}{2} + \frac{1}{4} e^{-2x}
\]

\[
 = \frac{1}{4} e^{2x} + \frac{1}{2} + \frac{1}{4} e^{-2x}
\]

\[
= \frac{1}{4} \left( e^{2x} + 2 + e^{-2x} \right)
\]

\[
= \frac{1}{2} \left( e^x + e^{-x} \right)^2
\]

\[
\sqrt{1 + (y')^2} = \frac{1}{2} \left( e^x + e^{-x} \right) \quad \text{and} \quad \left( \cosh x \right)
\]
\[ S = \int_0^2 \sqrt{1 + (y')^2} \, dy \]
\[ = \int_0^2 \frac{1}{2} (e^x + e^{-x}) \, dx = \frac{1}{2} \int_0^2 (e^x + e^{-x}) \, dx \]
\[ = \frac{1}{2} (e^x - e^{-x}) \bigg|_0^2 \]
\[ = \frac{1}{2} \left[ (e^2 - e^{-2}) - (1 - 1) \right] = \frac{1}{2} \left( e^2 - \frac{1}{e^2} \right) \]
\[ \approx 3.627 \]
8.1 Basic Integration Rules

(Rules listed on page 520)

Examples:

\[ \int \frac{2t - 1}{t^2 - t + 2} \, dt \]

Let \( u = t^2 - t + 2 \)
\[ du = (2t - 1) \, dt \]

\[ = \int \frac{du}{u} = \ln|u| + C \]
\[ = \ln|t^2 - t + 2| + C \]
### Procedures for Fitting Integrands to Basic Rules

<table>
<thead>
<tr>
<th>Technique</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>Expand (numerator).</td>
<td>$(1 + e^x)^2 = 1 + 2e^x + e^{2x}$</td>
</tr>
<tr>
<td>Separate numerator.</td>
<td>$\frac{1 + x}{x^2 + 1} = \frac{1}{x^2 + 1} + \frac{x}{x^2 + 1}$</td>
</tr>
<tr>
<td>Complete the square.</td>
<td>$\frac{1}{\sqrt{2x - x^2}} = \frac{1}{\sqrt{1 - (x - 1)^2}}$</td>
</tr>
<tr>
<td>Divide improper rational function</td>
<td>$\frac{x^2}{x^2 + 1} = 1 - \frac{1}{x^2 + 1}$</td>
</tr>
<tr>
<td>Add and subtract terms in numerator.</td>
<td>$\frac{2x^2 + 2 - 2}{x^2 + 2x + 1} = \frac{2x + 2}{x^2 + 2x + 1} - \frac{2}{(x + 1)^2}$</td>
</tr>
<tr>
<td>Use trigonometric identities.</td>
<td>$\cot^2 x = \csc^2 x - 1$</td>
</tr>
<tr>
<td>Multiply and divide by Pythagorean conjugate.</td>
<td>$\frac{1}{1 + \sin x} = \frac{1}{1 + \sin x} \left( \frac{1 - \sin x}{1 - \sin x} \right) = \frac{1 - \sin x}{1 - \sin^2 x}$</td>
</tr>
<tr>
<td></td>
<td>$\sin x = \frac{1 - \sin x}{\cos^2 x} = \sec^2 x - \frac{\sin x}{\cos^2 x}$</td>
</tr>
</tbody>
</table>
\[
\int \frac{1}{x\sqrt{x^2 - 4}} \, dx
\]

\[u = x, \quad du = dx\]

\[a = 2\]

\[\text{#20 } \int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} \arcsin \left| \frac{u}{a} \right| + C\]

\[\int \frac{1}{x\sqrt{x^2 - 4}} \, dx = \frac{1}{2} \arcsin \left| \frac{x}{2} \right| + C\]
\[ \int x \left( 1 + \frac{1}{x} \right)^3 \, dx \]

\[ = \int x \left( 1 + 3 \left( \frac{3}{x} \right)^2 + 3 \left( \frac{1}{x} \right)^2 + \frac{(1)}{x^3} \right) \, dx \]

\[ = \int x \left( 1 + \frac{3}{x} + \frac{3}{x^2} + \frac{1}{x^3} \right) \, dx \]

\[ = \int \left( x + 3 + \frac{3}{x} + \frac{1}{x^2} \right) \, dx \]

\[ = \frac{x^2}{2} + 3x + 3 \ln |x| + \frac{x^{-1}}{-1} + C \]

\[ = \frac{x^2}{2} + 3x + 3 \ln |x| - \frac{1}{x} + C \]
\[
\int \frac{1 + \cos \alpha}{\sin \alpha} \, d\alpha = \left( \frac{1}{\sin \alpha} + \frac{\cos \alpha}{\sin \alpha} \right) \, d\alpha \\
= \int \left[ \csc \alpha + \cot \alpha \right] \, d\alpha \\
= -\ln | \csc \alpha + \cot \alpha | + \ln | \sin \alpha | + C \\
= \ln \left| \frac{\sin \alpha}{\csc \alpha + \cot \alpha} \right| \\
= \ln \left| \frac{\sin \alpha}{\csc \alpha} (\csc \alpha - \cot \alpha) \right| \\
= \ln \left| \frac{\sin \alpha (\csc \alpha - \cot \alpha)}{\csc^2 \alpha - \cot^2 \alpha} \right| \\
= \ln \left| 1 - \cos \alpha \right| + C
\]
\[ \int_0^\pi \sin^2 t \cos t \, dt \]

Let \( u = \sin t \)

\( du = \cos t \, dt \)

\( u(0) = 0 \)

\( u(\pi) = 0 \)

\[ = \int_0^0 u^2 \, du = 0 \]

Or

\[ = \int_?^? u^2 \, du = \frac{u^3}{3} \bigg|_?^? = \frac{[\sin t]^3}{3} \bigg|_0^\pi \]

\[ = 0 - 0 = 0 \]
\[
\int_{1}^{e} \frac{1 - \ln x}{x} \, dx
\]

\[
0. \int_{1}^{e} \left[ \frac{1}{x} - \frac{\ln x}{x} \right] \, dx = \int_{1}^{e} \frac{1}{x} \, dx - \int_{1}^{e} \frac{\ln x}{x} \, dx
\]

\[
= \left[ \frac{e}{x} \right]_{1}^{e} - \int_{1}^{e} \frac{\ln x}{x} \, dx
\]

\[
= \ln|x| \left[ \frac{e}{x} \right]_{1}^{e} - \left[ \frac{u^2}{2} \right]_{0}^{1}
\]

\[
= [\ln e - \ln 1] - \frac{1}{2} = 1 - \frac{1}{2} = \left( \frac{1}{2} \right)
\]
\[
\int_{1}^{e} \frac{1 - \ln x}{x} \, dx
\]

\[= - \int_{1}^{e} (1 - \ln x) \left( -\frac{1}{x} \right) \, dx \]

\[= - \left[ u du = -u^2 \right]_{1}^{e} = - \left[ 0 - \frac{1}{2} \right] = \frac{1}{2}
\]

OR:

\[u = 1 - \ln x \]
\[du = -\frac{1}{x} \, dx\]
\[u(1) = 1 \quad ; \quad u(e) = 0\]
8.2 Integration by Parts

\[
\frac{d[f(x)g(x)]}{dx} = f'(x)g(x) + f(x)g'(x)
\]
\[\Rightarrow \int [f'(x)g(x) + f(x)g'(x)] \, dx = f(x)g(x)\]
\[\Rightarrow \int f'(x)g(x) \, dx + \int f(x)g'(x) \, dx = f(x)g(x)\]
\[\Rightarrow \text{solve for this integral}\]
\[\int f(x)g'(x) \, dx = f(x)g(x) - \int f'(x)g(x) \, dx\]

or
\[
\int uv \, du = uv - \int v \, du
\]
Theorem 8.1 Integration by Parts and Guidelines for Integration of Parts

THEOREM 8.1 Integration by Parts

If $u$ and $v$ are functions of $x$ and have continuous derivatives, then

$$\int u \, dv = uv - \int v \, du.$$

Guidelines for Integration by Parts

1. Try letting $dv$ be the most complicated portion of the integrand that fits a basic integration rule. Then $u$ will be the remaining factor(s) of the integrand.

2. Try letting $u$ be the portion of the integrand whose derivative is a function simpler than $u$. Then $dv$ will be the remaining factor(s) of the integrand.
Examples:

\[ \int x \ln x \, dx \]

Let \( u = \ln x \) and \( dv = x \, dx \),

\[ du = \frac{1}{x} \, dx \quad v = \frac{x^2}{2} \]

Then,

\[ \int u \, dv = uv - \int v \, du \]

\[ (\ln x)(x \, dx) = (\ln x) \left( \frac{x^2}{2} \right) - \int \frac{x^2}{2} \left( \frac{1}{x} \, dx \right) \]

\[ = \frac{x^2 \ln x}{2} - \frac{1}{2} \int x \, dx \]

\[ = \frac{2}{2} \cdot \frac{x^2 \ln x}{2} - \frac{1}{2} \cdot \frac{x^2}{2} + C \]

\[ = \frac{x^2 (2 \ln x - 1)}{4} + C \]
Check: \[ \frac{d}{dx} \left( \frac{x^2(2\ln x - 1)}{4} + C \right) \]

= \frac{1}{4} \left[ \frac{x^2(2\cdot\frac{1}{x}) + 2x(2\ln x - 1)}{x} \right]

= \frac{1}{4} \left[ 2x + 4x\ln x - 2x \right] = x\ln x \checkmark
\[ \int \ln x \, dx \]

\[ u = \ln x \quad dv = dx \]
\[ du = \frac{1}{x} \, dx \quad v = x \]

\[ \int u \, dv = uv - \int v \, du \]

\[ \int \ln x \, dx = (\ln x)(x) - \int x \cdot \frac{1}{x} \, dx \]

\[ = x \ln x - \int dx \]

\[ = x \ln x - x + C \]
\[
\int_0^1 x5^x \, dx = \int u \, dv - \int v \, du
\]

Let \( u = x \) and \( dv = 5^x \, dx \), then \( du = dx \) and \( v = \frac{5^x}{\ln 5} \).

\[
\int u \, dv = uv - \int v \, du
\]

\[
\int x(5^x) \, dx = x\left(\frac{5^x}{\ln 5}\right) - \int \frac{5^x}{\ln 5} \, dx
\]

\[
= x\frac{5^x}{\ln 5} - \frac{1}{\ln 5} \int 5^x \, dx
\]

\[
= x\frac{5^x}{\ln 5} - \frac{1}{\ln 5} \left(\frac{5^x}{\ln 5}\right)
\]

\[
\left[ x\frac{5^x}{\ln 5} - \frac{5^x}{(\ln 5)^2} \right]_0^1
\]

\[
= \left[ \frac{1(5)}{\ln 5} - \frac{5}{(\ln 5)^2} \right] - \left[ 0 - \frac{5^0}{(\ln 5)^2} \right]
\]

\[
= \left(\frac{\ln 5}{\ln 5} \right) \frac{5}{(\ln 5)^2} - \frac{5}{(\ln 5)^2} + \frac{1}{(\ln 5)^2}
\]

\[
= \frac{\ln 5 - 4}{(\ln 5)^2}
\]
Guidelines:
\[ u = \cos 2\theta \quad dv = e^{-\theta} d\theta \]

We will choose:
\[ u = e^{-\theta} \quad dv = \cos 2\theta d\theta \]
\[ w = 2\theta \quad dw = 2d\theta \]
\[ \int e^{-\theta} \cos 2\theta d\theta \]
\[ = \frac{1}{2} \int \cos w dw \]
\[ = \frac{1}{2} \sin w \Rightarrow \]
\[ v = \frac{1}{2} \sin (2\theta) \]
\[
\int e^{-\theta} \cos 2\theta \, d\theta
\]

\[
u = e^{-\theta} \quad dv = \cos 2\theta \, d\theta
\]
\[
u = -e^{-\theta} \quad v = \frac{1}{2} \sin 2\theta
\]

\[\int udv = uv - \int v \, du\]

\[
\int e^{-\theta} \cos 2\theta \, d\theta = e^{-\theta} \left(\frac{1}{2} \sin 2\theta\right) + \int \left(\frac{1}{2} \sin 2\theta\right) e^{\theta} \, d\theta
\]

Use integration by parts again.

\[
u = e^{-\theta} \quad dv = \sin 2\theta \, d\theta
\]
\[
u = -e^{-\theta} \quad v = -\frac{1}{2} \cos 2\theta
\]

\[
\int e^{-\theta} \cos 2\theta \, d\theta = \frac{1}{2} e^{-\theta} \sin 2\theta + \frac{1}{2} \left[ e^{-\theta} \left(\frac{1}{2} \cos 2\theta\right) - \int \left(\frac{1}{2} \cos 2\theta\right) (-e^{\theta}) \, d\theta\right]
\]

\[
\int e^{-\theta} \cos 2\theta \, d\theta = \frac{1}{2} e^{-\theta} \sin 2\theta - \frac{1}{4} e^{-\theta} \cos 2\theta - \frac{1}{4} e^{-\theta} \cos 2\theta
\]
\[
\begin{align*}
\int e^{-\theta} \cos 2\theta \, d\theta &= \frac{-1}{2} e^{-\theta} \sin 2\theta - \frac{1}{4} e^{-\theta} \cos 2\theta - \frac{1}{4} \int e^{-\theta} \cos 2\theta \, d\theta \\
&+ \frac{1}{4} \int e^{-\theta} \cos 2\theta \, d\theta \\
&= \frac{5}{4} \int e^{-\theta} \cos 2\theta \, d\theta = \frac{1}{2} e^{-\theta} \sin 2\theta - \frac{1}{4} e^{-\theta} \cos 2\theta \\
&\text{(multiply by } \frac{4}{5})
\end{align*}
\]

\[
\int e^{-\theta} \cos 2\theta \, d\theta = \frac{2}{5} e^{-\theta} \sin 2\theta - \frac{1}{5} e^{-\theta} \cos 2\theta + C
\]

**In general:**

\[
\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2 + b^2} \left[ b \sin bx + a \cos bx \right] + C
\]
\[ \int x^5 \cos(x^3) \, dx \quad w = x^3 \quad dw = 3x^2 \, dx \]

\[ = \frac{1}{3} \int x^3 \cos(x^3) \cdot (3x^2) \, dx \]

\[ = \frac{1}{3} \int w \cos(w) \, dw \quad u = w \quad dv = \cos(w) \, dw \]

\[ \quad du = dw \quad v = \sin(w) \]

\[ = \frac{1}{3} \left[ w \sin(w) - \int \sin(w) \, dw \right] \]

\[ = \frac{1}{3} \left[ w \sin(w) + \cos(w) \right] + C \]

\[ \int x^5 \cos(x^3) \, dx = \frac{1}{3} \int w \cos(w) \, dw \]

\[ = \frac{1}{3} \left[ w \sin(w) + \cos(w) \right] + C \]

\[ = \frac{1}{3} \left[ x^3 \sin(x^3) + \cos(x^3) \right] + C \]
(List of integration by parts integrals on page 530.

Skip Tabular Method.)

Homework: sections 8.1, 8.2