Section 3.1 (Homework)

#19. \( h(x) = 4x^2 - 4x + 21 \quad a = 4 \geq 0 \) opens up

\[
h(x) = 4\left(x^2 - x + \frac{1}{4}\right) + 21 - 1
\]

\[
h(x) = 4\left(x - \frac{1}{2}\right)^2 + 20 \quad \text{vertex } \left(\frac{1}{2}, 20\right)
\]

no x-intercepts

\[
\begin{array}{c|c|c}
  x & 3 & -3 \\
  \hline
  h(x) & 45 & 69
\end{array}
\]

- y-intercept \((0, 21)\)
\[ f(x) = \frac{-3}{5} (x^2 + 6x - 5) \quad a = \frac{-3}{5} < 0; \text{opens down} \]

\[ = \frac{-3}{5} (x^2 + 6x + 9) + 3 + \frac{27}{5} \]

\[ = \frac{15 - 27}{5} + \frac{27}{5} \]

\[ f(x) = \frac{-3}{5} (x+3)^2 + \frac{42}{5} \quad \text{vertex} \left( -3, \frac{42}{5} \right) \]

\[ \text{x-intercepts: set } f(x) = 0 = \frac{-3}{5} (x^2 + 6x - 5) \]

\[ \text{using QUAD: } (5.742, 0) \text{ and } (-6.742, 0) \]
3.3 Real Zeros of Polynomial Functions

Long Division of Polynomials

Example: \[
\frac{6x^3 - 16x^2 + 17x - 6}{3x - 2} = 2x^2 - 4x + 3
\]
The Division Algorithm

If \( f(x) \) and \( d(x) \) are polynomials such that \( d(x) \neq 0 \), and the degree of \( d(x) \) is less than or equal to the degree of \( f(x) \), there exist unique polynomials \( q(x) \) and \( r(x) \) such that

\[
f(x) = d(x)q(x) + r(x)
\]

where \( r(x) = 0 \) or the degree of \( r(x) \) is less than the degree of \( d(x) \). If the remainder \( r(x) \) is zero, \( d(x) \) divides evenly into \( f(x) \).

\[
\frac{f(x)}{d(x)} \text{ is improper.} \quad \frac{r(x)}{d(x)} \text{ is proper.}
\]
Example: Perform long division and write in the form $f(x) = d(x)q(x) + r(x)$.

1. \[ \frac{x^5 + 7}{x^3 - 1} \]

\[
\begin{array}{c}
\underline{x-1} \\
\overline{x^5} \\
\underline{0x^4} \\
\overline{0x^3} \\
\underline{0x^2} \\
-\underline{x^2} \\
\overline{0} \\
\end{array}
\]

\[ x^2 \]

\[ + x^2 + 7 \]

\[ x^5 + 7 = (x^3 - 1)(x^2) + x^2 + 7 \]
Synthetic Division (See pattern for dividing a cubic polynomial on page 279)

Synthetic Division is a short-cut process for dividing a polynomial of any degree by a polynomial of the form \( x - k \).

Example: Use synthetic division to divide.

2. \[
\frac{5x^3 + 18x^2 + 7x - 6}{x + 3}
\]

\[
\begin{array}{c|ccccc}
\multicolumn{1}{r|}{-3} & 5 & 18 & 7 & -6 \\
\hline
 & -15 & 9 & 6 \\
\hline
 & 5 & 3 & -2 & 10
\end{array}
\]

\[
\frac{5x^3 + 18x^2 + 7x - 6}{x + 3}
\]

\[
\begin{array}{c|ccccc}
\multicolumn{1}{r|}{-3} & 5 & 18 & 7 & -6 \\
\hline
 & -15 & 9 & 6 \\
\hline
 & 5 & 3 & -2 & 10
\end{array}
\]

Coefficients of quotient \( \Rightarrow -3 \) is a zero of the polynomial and \( (x + 3) \) is a factor
The Remainder Theorem
If a polynomial $f(x)$ is divided by $x - k$, the remainder is $r = f(k)$.

The Factor Theorem
A polynomial $f(x)$ has a factor $(x - k)$ if and only if $f(k) = 0$. 
Using the Remainder in Synthetic Division

In summary, the remainder $r$, obtained in synthetic division of $f(x)$ by $x - k$, provides the following information.

1. The remainder $r$ gives the exact value of $f$ at $x = k$. That is, $r = f(k)$.

2. If $r = 0$, $(x - k)$ is a factor of $f(x)$.

3. If $r = 0$, $(k, 0)$ is an $x$-intercept of the graph of $f$. 
Examples:

3. Use synthetic division and the Remainder Theorem to find $f(6)$ for $f(x) = 10x^4 - 50x^3 - 800$.

\[
\begin{array}{c|cccc}
6 & 10 & -50 & 0 & 0 \\
 & & 60 & 60 & 360 & 2160 \\
\hline
& 10 & 60 & 360 & 1360
\end{array}
\]

\[\Rightarrow f(6) = 1360\]
4. Show that \( x = -2 \) is a zero (or root) of
\[ f(x) = x^3 + 2x^2 - 2x - 4. \]
Factor completely and find all real zeros.

\[
\begin{array}{cccc}
-2 & 1 & 2 & -2 & -4 \\
 & -2 & 0 & +4 \\
 1 & 0 & -2 & 0 \Rightarrow f(-2) = 0 \Rightarrow \\
\hline
(x^2 + 2) is the other factor \\
(x + 2) is a factor \\
\end{array}
\]

\[ x^3 + 2x^2 - 2x - 4 = (x^2 + 2)(x + 2) \]

\[ = (x - \sqrt{2})(x + \sqrt{2})(x + 2) \]

real zeros: \( \pm \sqrt{2}, -2 \)
4. Show that $x = 2$ is a zero (or root) of $f(x) = x^3 + 2x^2 - 2x - 4$. Factor completely and find all real zeros.

$x = \sqrt{2}$

\[ \begin{array}{c|cc}
\sqrt{2} & 1 & 2 + -2 - 4 \\
\sqrt{2} & 2 + 2\sqrt{2} & 4 \\
\hline
1 & 2 + 2\sqrt{2} & 4 \\
\end{array} \]

$0 \Rightarrow \sqrt{2}$ is a zero

\[ x^2 + (2 + \sqrt{2})x + 2\sqrt{2} \]

\[ (x + 2)(x + \sqrt{2}) \] etc.
5. Use the Zero feature of your calculator to approximate the zeros of \( f(s) = s^3 - 12s^2 + 40s - 24 \) to three decimal places. Determine one of the exact zeros and use synthetic division to verify it. Factor completely.

\[
\begin{array}{c|cccc}
6 & 1 & -12 & 40 & -24 \\
\hline
 & 6 & -36 & 24 \\
 & \_ & -6 & 4 & 0 \\
\end{array}
\]

other factor is \( x^2 - 6x + 4 \)

use QUAD: \( x = 5.24, 0.76 \)

"looks about right"

zeros: 0.76, 5.24, 6
The Rational Zero Test

If the polynomial $f(x) = a_n x^n + \ldots + a_1 x + a_0$ has integer coefficients, every rational zero of $f$ has the form

$$\text{Rational zero} = \frac{p}{q}$$

Where $p$ and $q$ have no common factors other than 1, $p$ is a factor of the constant term $a_0$ and $q$ is a factor of the leading coefficient $a_n$. 
Examples: List all possible rational zeros.

6. \( f(x) = 4x^4 - 17x^2 + 4 \)

\[
a_n = a_4 = 4 \quad (1)(2)(4) \\
a_0 = 4 \quad (\pm 1)(\pm 2)(\pm 4)
\]

\[
\left(\frac{\text{factor of } a_0}{\text{factor of } a_n}\right) = \frac{\pm 1 \pm 1 \pm 1 \pm 2 \pm 2 \pm 4}{1 2 4 1 1 2 4} \\
\frac{\pm 4 \cdot \pm 4 \cdot \pm 4 }{1 1 1 4 4 4}
\]

same as \( \pm \frac{1}{1} \) etc.

\[
\pm 1, \pm \frac{1}{2}, \pm \frac{1}{4}, \pm 2, \pm 4
\]
7. \( f(x) = 6x^3 - x^2 - 13x + 8 \)

\[
\begin{cases}
a_n = a_3 = 6: 1, 2, 3, 6 \\
a_0 = 8: \pm 1, \pm 2, \pm 4, \pm 8
\end{cases}
\]

List possible rational zeros:

\[
\frac{p}{q} : \pm 1, \pm \frac{1}{2}, \pm \frac{1}{3}, \pm \frac{1}{6}
\]

(from \( a_0 \))

(from \( a_n \))

\[
\pm 2, \pm \frac{2}{1}, \pm \frac{2}{3}
\]

\[
\pm 4, \pm \frac{4}{1}, \pm \frac{4}{3}
\]

\[
\pm 8, \pm \frac{8}{1}, \pm \frac{8}{3}
\]
Skip Descartes' Rule of Signs and Upper and Lower Bounds.

Examples: Find all real zeros.

8. \( h(x) = -x^3 - 9x^2 + 20x - 12 \)

possible rational zeros: \( \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12 \)

Standard Screen

no real zeros \( \Rightarrow \) check \( \pm 12 \)

\[
egin{array}{c|cccc}
-12 & 1 & -9 & 20 & -12 \\
 & & 12 & -36 & 192 \\
 & & & -1 & -144 \\
\end{array}
\]

not a zero

\( h(x) = -x^3 - 9x^2 + 20x - 12 \)
9. \( f(z) = 12z^3 - 4z^2 - 27z + 9 \)

Possible rational zeros:

\[
\begin{array}{cccc}
1 & 3 & 12 & -4 & -27 & 9 \\
 & 4 & 0 & -9 \\
 & 12 & 0 & -27 & 0 \\
\end{array}
\]

\( \frac{1}{3} \) is a zero
\( (x - \frac{1}{3}) \) is a factor

Other factor is \( 12x^2 - 27 \)

Set \( 12x^2 - 27 = 0 \)

\[ x^2 = \frac{27}{12} = \frac{9}{4} \]

\[ x = \pm \frac{3}{2} \]