Section 2.3 continued

Definition:
The pair of complex numbers \( a+bi \) and \( a-bi \) are called **complex conjugates**.

Note: \( (a+bi)(a-bi) = a^2 + b^2 \), a real number
\[ a^2 - ab + abi = a^2 + b^2 (-1) = a^2 + b^2 \]

We use conjugates to find the quotient of two complex numbers.
\[
\frac{11-2i}{-3+6i} \cdot \frac{-3-6i}{-3-6i} = \frac{-45-60i}{9+36} = \frac{-45-60i}{45} = -1 - \frac{4}{3} i
\]
or, using a calculator,
\[
\frac{(11-2i)}{(-3+6i)} = -1-1.33333333333i
\]
Ans\(\uparrow\) Frac
\[
-1-4/3i
\]
Examples: Find the quotient.  \( \frac{32i}{-1} \)

8. \[
\frac{-8i}{9 + 4i} \cdot \frac{9 - 4i}{9 - 4i} = \frac{-72i - 32i^2}{9^2 + 4^2} = \frac{-32 - 72i}{81 + 16} = \frac{-32}{97} - \frac{72}{97}i
\]

9. \[
\frac{9 + 4i}{-8i} \cdot \frac{8i}{8i} = \frac{72i + 32i^2}{64} = -\frac{32}{64} + \frac{22}{64}i
= -\frac{1}{2} + \frac{9}{8}i
\]
\[ i^1 = i, i^2 = -1, i^3 = -i, i^4 = 1 \]

Powers of \( i \):
\[ i^5 = i, i^6 = -1, i^7 = -i, i^8 = 1 \text{ etc.} \]

\[ i^3 = i^2 \cdot i = (-1) \cdot i = -i \]

\[ i^4 = i^2 \cdot i^2 = (-1)(-1) = 1 \]

\[ i^5 = i^4 \cdot i = 1 \cdot i = i \]

\[ i^6 = i^4 \cdot i^2 = 1 \cdot (-1) = -1 \]

\[ i^7 = i^6 \cdot i = -i \]

\[ i^8 = i^4 \cdot i^4 = 1 \cdot 1 = 1 \]

\[ i^9 = i^8 \cdot i = 1 \cdot i = i \]

\[ \sqrt{\frac{2}{9}} = \frac{\sqrt{2}}{3} \]
Example: Write as $i$, $-1$, $-i$, or $1$:

\[ i^{37} = (i^4)^9 \cdot 1 = 1 \cdot i = i \]

\[ i^{37} = (\frac{9}{36}) \cdot 1 = \frac{9}{36} \]

\[ i^{62} = (i^4)^{15} \cdot 2 = 1 (-1) = -1 \]

\[ i^{62} = (\frac{15}{60}) \cdot 2 = \frac{15}{60} \cdot 2 = \frac{3}{2} \]

\[ i^{47} = (i^4)^{11} \cdot 3 = 1 \cdot (-i) = -i \]

\[ i^{47} = (\frac{11}{44}) \cdot 3 = \frac{11}{44} \cdot 3 = \frac{33}{44} \cdot 3 = \frac{3}{3} \]
Skip Fractals and the Mandelbrot Set, but look at Example 6, *Plotting Complex Numbers.*

**Definition:**
The **magnitude** of a complex number \( a + bi \) is \( \sqrt{a^2 + b^2} \) and is the distance from the origin to \( a + bi \) on the complex plane.
Examples:
11. Plot each of the following complex numbers: 
   \(3 + 2i, -4i, -2 - 5i\) and find the magnitude of each.
Every real number has \( n \)th roots in the complex number system.

\[ x^3 = 8 \quad x = 2 \]

\[
(-1 + \sqrt{3}i)^3 = (-1 + \sqrt{3}i)(-1 + \sqrt{3}i)(-1 + \sqrt{3}i)
\]

\[
= (1 - 2\sqrt{3}i - 3)(-1 + \sqrt{3}i)
\]

\[
= (-2 - 2\sqrt{3}i)(-1 + \sqrt{3}i)
\]

\[
= (2 - 2\sqrt{3}i + 2\sqrt{3}i - 2(3)i^2)
\]

\[
= 2 + 6 = 8
\]
2.4 Solving Quadratic Equations Algebraically

Definition
A **quadratic equation in** \( x \) **is an equation** that can be written in the standard form \( ax^2 + bx + c = 0 \) **where** \( a, b, \) and \( c \) **are real numbers** and \( a \neq 0 \). (Also called a **second-degree polynomial equation in** \( x \).)
Solving a Quadratic Equation:

A. *Factoring* Use the Zero-Factor Property: If \( ab = 0 \), then \( a = 0 \) or \( b = 0 \) or both are 0.

Examples: Solve by factoring.

1. \( 9x^2 - 1 = 0 \)

\[
(3x-1)(3x+1) = 0
\]

\[
\begin{align*}
3x-1 &= 0 & 3x+1 &= 0 \\
3x &= 1 & 3x &= -1 \\
x &= \frac{1}{3} & x &= -\frac{1}{3}
\end{align*}
\]
2. \( x^2 - 10x + 9 = 0 \)

\[ (x-9)(x-1) = 0 \]

\[ x-9 = 0 \quad \text{or} \quad x-1 = 0 \]

\[ x = 9 \quad \text{or} \quad x = 1 \]

Check: \( 9^2 - 10(9) + 9 = 81 - 90 + 9 = 0 \) \( \checkmark \)

\( 1^2 - 10(1) + 9 = 0 \) \( \checkmark \)

3. \( 2x^2 = 19x + 33 \)

\[ 2x^2 - 19x - 33 = 0 \]

\[ (2x+3)(x-11) = 0 \]

\[ 2x+3 = 0 \quad \text{or} \quad x-11 = 0 \]

\[ x = -\frac{3}{2} \quad \text{or} \quad x = 11 \]
B. Extracting Square Roots

If \( u^2 = c \), and \( c > 0 \), then
\[
u = \pm \sqrt{c}.
\]
(use this method if there is no "x" term)

Examples: Solve by extracting square roots.

4. \( 9x^2 = 25 \)
\[
\iff x = \frac{25}{9} \iff x = \pm \sqrt{\frac{25}{9}} = \pm \frac{5}{3}
\]

5. \( 9x^2 - 25 = 0 \)
\[
\iff 3x - 5 = 0 \iff x = \frac{5}{3}
\]
\[
\iff 3x + 5 = 0 \iff x = -\frac{5}{3}
\]

5. \( (x - 5)^2 = 20 \)
\[
\iff x - 5 = \pm \sqrt{20}
\]
\[
\iff x = 5 \pm \sqrt{20} \]
\[
\iff x = 5 \pm \sqrt{4.5}
\] 

exact form

approximate form

\[
\begin{array}{c}
5 - 2\sqrt{5} \approx 5.27864045 \\
5 + 2\sqrt{5} \approx 9.472135955
\end{array}
\]
Completing the Square:

\( ax^2 + bx + c = 0 \)

\( x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \)

\[ x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2} \left( \frac{1}{2} \frac{b}{a} \right)^2 \text{ add to both sides} \]

\[ (x + \frac{b}{2a})^2 = \frac{b^2 - 4ac}{4a^2} \]

\[ (x + \frac{b}{2a}) = \pm \sqrt{\frac{b^2 - 4ac}{4a^2}} \]

\[ x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \]
Examples: Solve by completing the square.

6. \(x^2 + 8x + 14 = 0\)

\[
\begin{align*}
  x^2 + 8x & = -14 \\
  x^2 + 8x + 16 & = -14 + 16 \\
  (x + 4)^2 & = 2 \\
  x + 4 & = \pm \sqrt{2} \\
  x & = -4 \pm \sqrt{2} \\
  x & = -5.414, -2.586 \text{ (approx)}
\end{align*}
\]
7. \[ 4x^2 - 4x - 99 = 0 \]

\[
\begin{align*}
X^2 - X - \frac{99}{4} &= 0 \\
X^2 - X + \frac{1}{4} &= \frac{99}{4} + \frac{1}{4} \\
\left(X - \frac{1}{2}\right)^2 &= \frac{100}{4} = 25 \\
X - \frac{1}{2} &= \pm \sqrt{25} \\
X &= \frac{1}{2} \pm 5
\end{align*}
\]

\[ \text{add} \left(\frac{1}{2}\right)^2 = \frac{1}{4} \]

\[ \frac{1}{2} \left(\frac{1}{2}\right) = \frac{1}{4} \]

\[ \frac{1}{2} \left(\frac{1}{2}\right) \to \text{both sides} \]

\[ \left(X - \frac{1}{2}\right)^2 = \frac{100}{4} = 25 \]

\[ X - \frac{1}{2} = \pm \sqrt{25} \]

\[ X = \frac{1}{2} + 5 = \frac{11}{2} \]

\[ X = \frac{1}{2} - 5 = -\frac{9}{2} \]
D. Quadratic Formula  If $ax^2 + bx + c = 0$, then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
Examples: Solve by using the quadratic formula.

8. \( x^2 - 10x + 22 = 0 \)

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

\[
x = \frac{-(-10) \pm \sqrt{(-10)^2 - 4(1)(22)}}{2(1)} = \frac{10 \pm \sqrt{100 - 88}}{2}
\]

\[
x = \frac{10 \pm \sqrt{12}}{2} = \frac{10 \pm \sqrt{4 \cdot 3}}{2} = \frac{10 \pm 2\sqrt{3}}{2}
\]

\[
x = \frac{5 \pm \sqrt{3}}{2} = \left(\frac{5 \pm \sqrt{13}}{2}\right)	ext{ exact form}
\]
9. \(9x^2 + 24x + 16 = 0\)

\[
\left(3x + 4\right)^2 = 0 \iff 3x + 4 = 0 \quad (x = -\frac{4}{3})
\]

\[
\left(3x + 4\right)\left(3x + 4\right)
\]

Double root (repeated root)

\(x = -\frac{4}{3}\) is a zero of multiplicity 2

\[
b^2 - 4ac = 24^2 - 4(9)(16) = 0\]

(discriminant)
10. \[9x^2 - 6x - 35 = 0\] 
\[a = 9; b = -6; c = -35\]

\[x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\]

\[= \frac{-(-6) \pm \sqrt{(-6)^2 - 4(9)(-35)}}{2(9)} = \frac{6 \pm \sqrt{36 + 1260}}{18}\]

\[= \frac{6 \pm \sqrt{1296}}{18} = \frac{6 \pm 36}{18}\]

\[x = \frac{6 + 36}{18} = \frac{42}{18} = \frac{7}{3}\]

\[x = \frac{6 - 36}{18} = \frac{-30}{18} = -\frac{5}{3}\]
11. $x^2 + 2x + 3 = 0$  \( a=1; b=2; c=3 \)

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

\[
= \frac{-2 \pm \sqrt{2^2 - 4(1)(3)}}{2(1)} = \frac{-2 \pm \sqrt{-8}}{2} = \frac{-2 \pm \sqrt{4(-2)}}{2}
\]

\[
= -2 \pm \frac{2i\sqrt{2}}{2} = \frac{-2 \pm 2i\sqrt{2}}{2}
\]

\[
= -1 \pm i\sqrt{2}
\]
Definition:
For the quadratic equation $ax^2 + bx + c = 0$, the quantity $b^2 - 4ac$ is called the discriminant.

If $b^2 - 4ac > 0$, there are two different real zeros.
If $b^2 - 4ac = 0$, there is one repeated real zero.
If $b^2 - 4ac < 0$, there are two complex conjugate zeros.
Example:

11α) The distance \( d \) (in miles) a car can travel on one tank of fuel is approximated by
\[
d = -0.024s^2 + 1.455s + 431.5, \quad 0 < s \leq 75,
\]
where \( s \) is the average speed of the car in miles per hour.
a. Use a graphing utility to graph the function over the specified domain.

\[ y_1 = -0.024x^2 + 1.45 \]
\[ 5x + 431.5 \]

**WINDOW**
- Xmin = 0
- Xmax = 75
- Xscl = 1
- Ymin = -10
- Ymax = 10
- Yscl = 1
- Xres = 1

**ZOOM MEMORY**
- 4: ZDecimal
- 5: ZSquare
- 6: ZStandard
- 7: ZTrig
- 8: ZInteger
- 9: ZoomStat
- 10: ZoomFit

- total distance
- max
- speed

Maximum: 
- X = 30.312491
- Y = 453.552341
b. Use the graph to determine the greatest distance that can be traveled on a tank of fuel. How long will the trip take?

\[ d = rt \]

\[ t = \frac{d}{r} \]

453.6 miles (maximum distance) occurred when \( x(s) = 30.3 \text{ mph} \)

\[ \frac{453.6 \text{ miles}}{30.3 \text{ miles/hour}} = 15.0 \text{ hours} \]
c. Determine the greatest distance that can be traveled in this car in 8 hours with no refueling. How fast should the car be driven? [Hint: The distance traveled in 8 hours is 8s. Graph this expression in the same viewing window as the graph in part (a) and approximate the point of intersection.

439.1 miles with average speed 54.9 miles/hour

Homework Sections 2.3 and 2.4