2.1 Linear Equations and Problem Solving

Definitions:
An equation in \( x \) is a statement that two algebraic expressions are equal.
To solve an equation in \( x \) means to find all values of \( x \) for which the equation is true.
Values of \( x \) for which an equation is true are called its solutions.
An equation that is true for every real number in the domain of the variable is called an identity.
An equation that is true for just some (or even none) of the real numbers in the domain of the variable is called a conditional equation.
An equation which is not true for any real number is called a contradiction.
A linear equation in one variable $x$ is an equation that can be written in the standard form $ax + b = 0$ where $a$ and $b$ are real numbers, with $a \neq 0$.
An extraneous solution is one that does not satisfy the original equation. It is often introduced when an equation is multiplied or divided by a variable expression.
Examples: Solve the following equations. Use a graphing utility to verify your solutions.

\[ \text{LCM} = 10 \]
\[ \frac{x}{5} - \frac{x}{2} = 3 \text{ (multiply by 10)} \]
\[ 2 - 5x = 30, -3x = 30 \]
\[ x = -10 \]
2. \[ \frac{17 + y}{y} + \frac{32 + y}{y} = 100 \] (by y)

\[ y \left(\frac{17+y}{y}\right) + y \left(\frac{32+y}{y}\right) = 100 y \]

\[ 17 + y + 32 + y = 100 y \]
\[ 49 + 2y = 100 y \]
\[ 49 = 98 y \]
\[ \frac{1}{2} = y \]
3. \[ \frac{x}{x+4} + \frac{4}{x+4} + 2 = 0 \] (by \( x+4 \))

\[(x+4) \left( \frac{x}{x+4} \right) + (x+4) \left( \frac{4}{x+4} \right) + 2(x+4) = 0(x+4)\]

\[x + 4 + 2x + 8 = 0\]

\[3x + 12 = 0\]

\[3x = -12\]

\[x = -4\] extraneous solution

no solution
Using algebra (or other mathematics) to solve problems that occur in real-life situations is called **mathematical modeling**.

See the list of common formulas on page 166.

Examples: Solve for the indicated variable.

4. *Investment at Simple Interest*

Solve for \( r \): \( A = P + P \cdot r \cdot t \)

\[
\frac{A - P}{P \cdot t} = \frac{r \cdot t}{P \cdot t}
\]

\[
\frac{A - P}{P \cdot t} = r
\]

**STOP!**
5. Volume of a Spherical Segment

Solve for \( r \):

\[
V = \frac{1}{3} \pi h^2 (3r - h)
\]

(mult. by 3)

\[
3V = 3\pi h^2 r - \pi h^3
+ \pi h^3
\]

\[
\frac{3V + \pi h^3}{3\pi h^2} = \frac{3\pi h^2 r}{8\pi h^2}
\]

\[
\frac{3V + \pi h^3}{3\pi h^2} = r
\]

STOP!
Examples:  (from pp. 169–170)

12. A picture frame has a total perimeter of 3 meters. The height of the frame is $\frac{2}{3}$ times its width.
   a. Draw a picture that gives a visual representation of the problem. Identify the width as $w$ and the height as $h$. 

\[ h \]

\[ w \]
b. Write $h$ in terms of $w$ and write an equation for the perimeter in terms of $w$.

\[
h = \frac{2}{3}w \quad \frac{4}{3}w + \frac{1}{3}w
\]

\[
P = 2h + 2w = 2\left(\frac{2}{3}w\right) + 2w = \frac{10}{3}w \Rightarrow P = \frac{10}{3}w
\]

c. Find the dimensions of the picture frame.

\[
3 = \frac{10}{3}w; \quad w = 3\left(\frac{3}{10}\right) = \frac{9}{10}
\]

\[
h = \frac{2}{3}\left(\frac{9}{10}\right) = \frac{3}{5}
\]

\[
w = \frac{9}{10} \text{ meters and } h = \frac{3}{5} \text{ meters}
\]

\[
\begin{array}{c}
9/10 \\
\hline
6/10
\end{array}
\]

\[
\frac{3}{5} = \frac{6}{10}
\]

\[
w = \frac{9}{10}
\]

\[
\frac{9}{10} + \frac{6}{10} + \frac{6}{10} = \frac{30}{10} = 3
\]
from Section 1.6 #78.

\[ r = \frac{1}{2} x \quad \Rightarrow \quad r(x) = \frac{1}{2} x \]

\[ A = \pi r^2 \]

\[(A \circ r)(x) = A(r(x)) = \pi \left[ r(x) \right]^2 \]

\[ = \pi \left[ \frac{1}{2} x \right]^2 = \frac{1}{4} \pi x^2 \]

**Example:**

If \( x = 10 \), then \( r = 5 \)

\[ A = \pi r^2 = \pi (5)^2 = 25 \pi \]

\[ A = \frac{1}{4} \pi x^2 = \frac{1}{4} \pi (10)^2 = \frac{1}{4} \pi (100) = 25 \pi \]

(area of circle as a function of \( x \))
1. (4% each) Name the type of function represented by each of the following graphs:

a. square root
b. cubic
c. quadratic

a. absolute value
b. identity
c. constant
2. (4% each)

a. Determine the equation of the line that passes through \((-2, 4)\) with a slope of 3.

\[
y - y_1 = m(x-x_1)
\]
\[
y - 4 = 3(x + 2)
\]
\[
y - 4 = 3(x + 2)
\]
\[
y - 4 = 3(x - (-2))
\]
\[
y - 4 = 3(x - (-2))
\]
\[
y - 4 = 3x + 6
\]
\[
y = 3x + 10
\]
\[
0 = 3x - y + 10
\]

b. Determine the slope of the line that passes through \((-1, 5)\) and \((3, 3)\).

\[
m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3 - 5}{3 - (-1)} = \frac{-2}{4} = -\frac{1}{2}
\]

\[
b. \quad -\frac{1}{2}
\]
c. Sketch: \( f(x) = \begin{cases} \frac{x^2}{2} & x \leq 1 \\ -x & x > 1 \end{cases} \)
3. (4% each) For the following graphs:

a. Which graphs represent $y$ as a function of $x$? $\boxed{\text{b and c}}$

b. Which graphs represent $y$ as one-to-one functions of $x$? $\boxed{\text{c}}$

Vertical line test

Horizontal line test
4. (4% each) Determine whether each of the following is even, odd, or neither.

\[ f(x) = -x^2 \]
\[ f(-x) = -(-x)^2 = -x^2 = f(x) \]
\[ \text{even} \]

\[ f(x) = 3 \]
\[ f(-x) = 3 = f(x) \]
\[ \text{even} \]

\[ f(x) = -x \]
\[ f(-x) = -(-x) = x = -f(x) \]
\[ \text{odd} \]

\[ f(x) = x^2 + x \]
\[ f(-x) = (-x)^2 + (-x) = x^2 - x \]
\[ \text{neither} \]
5. Describe the transformation of \( f(x) = \sqrt{x} \) for the graph of \( g(x) = -\sqrt{x+1} - 2 \).

a. **Horizontal shift 1 unit to left** \( (\sqrt{x+1}) \)

b. **Reflection in x-axis** \( (-\sqrt{x+1}) \)

c. **Vertical shift 2 units down** \( (-\sqrt{x+1} - 2) \)
6. (4% each) Find the functions represented by the following, start with \( f(x) = x^3 \)

\[ f(x) = x^3 \]

a. Reflect \( f(x) \) about the \( x \)-axis.
The new function \( g(x) = \)

\[ -x^3 \]

b. Shift \( g(x) \) 2 units to the left.
The new function \( h(x) = \)

\[ - (x+2)^3 \]

c. Sketch \( h(x) \) vertically by a factor of 3.
The new function \( w(x) = \)

\[ 3[-(x+2)^3] \]
7. Given: \( f(x) = x^2 \) and \( g(x) = x + 2 \). Find:

a. \((f \circ g)(x) = \frac{f(g(x))}{g(x)} = (g(x))^2 = (x+2)^2\)

b. \(f(3) + g(4) = 3^2 + (4+2) = 9 + 6\)

c. The domain of \(\frac{f}{g}(x)\) is \(x \neq -2\).
8. \( f(x) = 4x^2 - x^4 \), find:

(Hint: Use a window with \(-10\) ≤ \(x\) ≤ \(10\), \(-10\) ≤ \(y\) ≤ \(10\))

a. Any relative maxima.

b. Any relative minima.
c. Those intervals where $f$ is increasing.
\[ (-\infty, -1.414) \text{ and } (0, 1.414) \]

(intervals with respect to $x$)

d. Those intervals where $f$ is decreasing.
\[ (-1.414, 0) \text{ and } (1.414, \infty) \]
9. (4%) Find the inverse of \( f(x) = \frac{1}{2} x + 3 \).

Let \( y = f(x) \)

\[ y = \frac{1}{2} x + 3 \]

Interchange \( x \) and \( y \)

\[ x = \frac{1}{2} y + 3 \quad \text{(`by 2}) \]

Solve for \( y \)

\[ 2x = y + 6 \]

\[ 2x - 6 = y \]

The new \( y \) is \( f^{-1}(x) \)

\[ f^{-1}(x) = 2x - 6 \]

Check: \( f^{-1}(f(x)) = 2f(x) - 6 \)

\[ = 2 \left( \frac{1}{2} x + 3 \right) - 6 = x + 6 - 6 = x \]
10. The population $y$ in a certain city between 1980 and 2010 can be modeled by
$y = 4x^4 + 5x^3 - 15x^2 + 1020x + 10000$ where $x = 0$ corresponds to 1980. Use your
calculator to estimate the population in 2002.

$y = 4x^4 + 5x^3 - 15x^2 + 1020x + 10000$

$x = 0$ corresponds to 1980.

Estimate population in 2002.

$x = 22$

$y = 4(22)^4 + 5(22)^3 - 15(22)^2 + 1020(22) + 10000$

or Table.