Section 1.6 continued

Definition: The composition of the function $f$ with the function $g$ is $(f \circ g)(x) = f(g(x))$. The domain of $f \circ g$ is the set of all $x$ in the domain of $g$ such that $g(x)$ is in the domain of $f$.

Examples: In each of the following, find $(f \circ g)(x), (g \circ f)(x), (f \circ f)(x)$. (Problems from p. 142)

36. $f(x) = \sqrt[3]{x-1}, g(x) = x^3 + 1$

$$(f \circ g)(x) = f(g(x)) = \sqrt[3]{g(x)-1}$$

$$= \sqrt[3]{x^3 + 1 - 1} = \sqrt[3]{x^3} = x \neq 0$$

$$(g \circ f)(x) = g(f(x)) = \left[ f(x) \right]^3 + 1$$

$$= \left[ \sqrt[3]{x-1} \right]^3 + 1 = x - 1 + 1 = x$$

$$(f \circ f)(x) = f(f(x)) = \sqrt[3]{f(x)-1}$$

$$= \sqrt[3]{\sqrt[3]{x-1}-1}$$
\((f \circ g)(x)\), \((g \circ f)(x)\), \((f \circ f)(x)\)

38. \(f(x) = x^3, \quad g(x) = \frac{1}{x}, \quad x \neq 0\)

\((f \circ g)(x) = f(g(x)) = \left(\frac{1}{x}\right)^3 = \frac{1}{x^3}, \quad x \neq 0\)

\((g \circ f)(x) = g(f(x)) = \frac{1}{x^3}, \quad x \neq 0\)

\(f \circ f(x) = f(f(x)) = [f(x)]^3 = [x^3]^3 = x^9\)
\((f \circ g)(x)\); \((g \circ f)(x)\)

40. \(f(x) = \sqrt{x+3}\) \(g(x) = \frac{x}{2}\)

\((f \circ g)(x) = f(g(x)) = \sqrt{g(x)+3} = \sqrt{\frac{x}{2}+3}\)

\[= \sqrt{\frac{x+6}{2}}\]

\((g \circ f)(x) = g(f(x)) = \frac{f(x)}{2} = \frac{\sqrt{x+3}}{2}\)
52. Use the graphs of \( f \) and \( g \) to evaluate \( (a.) \ (f-g)(1) \) and \( (b.) \ (fg)(4) \).

\[ \text{Graphs of } f \text{ and } g \]

\[ a.) \ (f-g)(1) = f(1) - g(1) = 2 - 3 = -1 \]

\[ b.) \ (fg)(4) = (f(4))(g(4)) = (4)(0) = 0 \]
66.56. Find two functions \( f \) and \( g \) such that 
\[ (f \circ g)(x) = h(x) \] if \( h(x) = (1-x)^3 \).

\[
\begin{align*}
g(x) &= 1-x \\
f(x) &= x^3 \\
h(x) &= f(g(x)) = (g(x))^3 \\
 &= (1-x)^3
\end{align*}
\]

Let \( w(x) = \sqrt{x+2} \); find 2 functions \( f \) and \( g \) such that \( w(x) = (f \circ g)(x) \).

1. Let \( g(x) = x+2 \) \( \Rightarrow \) \( w(x) = (f \circ g)(x) = f(g(x)) \)

\[
\begin{align*}
g(x) &= x+2 \\
f(x) &= \sqrt{x} \\
w(x) &= f(g(x)) = f(x+2) = \sqrt{x+2}
\end{align*}
\]

2. Let \( f(x) = \sqrt{x} \) \( \Rightarrow \) \( (f \circ g)(x) = f(g(x)) \)

\[
\begin{align*}
f(x) &= \sqrt{x} \\
g(x) &= x+2 \\
\frac{x+2}{2} &= \frac{x+2}{2}
\end{align*}
\]

Note: \( x+2 \neq x+1 \)

\[
\frac{2x+2}{2} = \frac{2(x+1)}{2} = x+1
\]
1.7 Inverse Functions

Definition: Let $f$ and $g$ be two functions such that
\[ f(g(x)) = x \] for every $x$ in the domain of $g$, and
\[ g(f(x)) = x \] for every $x$ in the domain of $f$.
Under these conditions, the function $g$ is the inverse function of the function $f$.

The function $g$ is denoted $f^{-1}$ (read “$f$-inverse”).

Therefore
\[ f(f^{-1}(x)) = x \quad \text{and} \quad f^{-1}(f(x)) = x. \]

The domain of $f$ must be equal to the range of $f^{-1}$ and
the range of $f$ must be equal to the domain of $f^{-1}$. 

\[ \text{Note:} \quad f^{-1} \neq \frac{1}{f} \]
In a previous section, we looked at the functions:

\[ f(x) = \sqrt[3]{x} - 1 \quad \text{and} \quad g(x) = x^3 + 1 \quad \text{and found that} \]

\[ f(g(x)) = x, g(f(x)) = x \quad \text{implying that} \ f \ \text{and} \ g \ \text{are inverses.} \]

The following shows the graph of \( y = f(x), \ y = g(x), \) and \( y = x. \)

(Note that this is a Square screen.)

The graphs of \( f \) and \( g \) are symmetric in the line \( y = x. \) This is true for any pair of inverse functions.
Examples: Using common sense, find the inverses of the following functions:

\[ f(x) = x + 2 \quad \Rightarrow \quad f^{-1}(x) = x - 2 \]

\[ f(f^{-1}(x)) = f^{-1}(x) + 2 = x - 2 + 2 = x \quad \checkmark \]

\[ f(x) = \frac{x}{3} \quad \Rightarrow \quad f^{-1}(x) = 3x \]

\[ f^{-1}(f(x)) = 3f(x) = 3 \left( \frac{x}{3} \right) = x \quad \checkmark \]
\[ g(x) = x^2 \]

\[ \Rightarrow g(x) \text{ as is does not have an inverse function} \]

What we will do is to restrict the domain of \( g \):

\[ g(x) = x^2, \quad x \geq 0 \]

\text{domain } x \geq 0; \text{ range } y \geq 0

\text{Guess: } g^{-1}(x) = \sqrt{x}\]

\text{domain } x \geq 0; \text{ range } y \geq 0
Definition: A function \( f \) is \textbf{one-to-one} if, for \( a \) and \( b \) in its domain, \( f(a) = f(b) \) implies that \( a = b \).

A function \( f \) has an inverse function \( f^{-1} \) if and only if \( f \) is one-to-one.
Horizontal Line Test: If every horizontal line intersects the graph of a function $f$ at most once, then the function is one-to-one. (That is, no horizontal line intersects the graph of the function more than once.)

**Examples:**

- $y_1 = 2x + 1$
  
  - **One-to-one function**
  
  - **Horizontal line test:** passes both
  
  - **Vertical line test:** passes

- $y_2 = x^2$

  - **Function, not one-to-one**
  
  - **Horizontal line test:** fails VLT
  
  - **Vertical line test:** passes HLT

- Not a function

  - **Horizontal line test:** fails VLT
Finding an inverse function:

1. Use the Horizontal Line Test to decide whether $f$ has an inverse function.

2. In the equation for $f(x)$, replace $f(x)$ with $y$.

3. Interchange the roles of $x$ and $y$, and solve for $y$.

4. Replace $y$ by $f^{-1}$ in the new equation.

5. Verify that $f$ and $f^{-1}$ are inverse functions by showing that the domain of $f$ is equal to the range of $f^{-1}$ and the range of $f$ is equal to the domain of $f^{-1}$ and that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$. 

Examples: In each case, determine if the function is one-to-one. If so, find its inverse.

\[ f(x) = 3x + 5 \]

\[ f(x) = 3x + 5 \]

Replace \( f(x) \) with \( y \)

Interchange \( x \) and \( y \)

Solve for \( y \)

This new \( y = f^{-1} \)

Also, find \( f(f^{-1}(x)) \):

\[ f(f^{-1}(x)) = 3f^{-1}(x) + 5 \]

\[ = 3\left(\frac{x-5}{3}\right) + 5 \]

\[ = x - 5 + 5 = x \]
52. \[ h(x) = \frac{4}{x^2} \]

Clearly fails the horizontal line test \( \Rightarrow \) it is not one-to-one \( \Rightarrow \) it has no inverse
54. \( q(x) = (x - 5)^2 \)  

- Domain: \( x \leq 5 \)  
- Range: \( y \geq 0 \)  

The function is one-to-one \( \Rightarrow \) it has an inverse.

Replace \( f(x) \) with \( y \): 
\[ y = (x - 5)^2 \]

Interchange \( x \) and \( y \): 
\[ x = (y - 5)^2 \]

Solve for \( y \): 
\[ \pm \sqrt{x} = y - 5 \Rightarrow \sqrt{x} + 5 = y \]
\[ y = \pm \sqrt{x} + 5 \]

- Domain: \( x \geq 0 \)  
- Range: \( y \leq 5 \)  

The new function \( f^{-1} \):
\[ f^{-1}(x) = -\sqrt{x} + 5 \]  

Also: \( (f^{-1})'(x) = \frac{1}{2\sqrt{x}} \)  
\[ f'(f(5)) = x \]
\[ f'(f(5)) = -\sqrt{f(5)} + 5 \%
\[ = -\sqrt{(x - 5)^2} + 5 \]
\[ = -|x - 5| + 5 \]
\[ = -|x - 5| + 5 \]
\[ = - (x - 5) + 5 = x - 5 + 5 = x \]  

Homework: Sections 1.6 and 1.7  
Test #2 Tuesday, October 16
2.1 Linear Equations and Problem Solving

Definitions:
An **equation** in $x$ is a statement that two algebraic
equations are equal.
To **solve** an equation in $x$ means to find all values of $x$
for which the equation is true.
Values of $x$ for which an equation is true are called its
**solutions**.
An equation that is true for every real number in the
domain of the variable is called an **identity**.
An equation that is true for just some (or even none) of
the real numbers in the domain of the variable is called
a **conditional equation**.

**Examples:**

- $x + 3 = 3 + x$, true for all $x$, **Identity**
- $x + 3 = 2$, true for $x = -1$, **Conditional**
- $x + 3 = x + 2$, never true, **Contradiction**
An equation which is not true for any real number is called a **contradiction**.  
A **linear equation in one variable** $x$ is an equation that can be written in the standard form $ax + b = 0$ where $a$ and $b$ are real numbers, with $a \neq 0$.  
An **extraneous solution** is one that does not satisfy the original equation. It is often introduced when an equation is multiplied or divided by a variable expression.