Test #3

#5b. \( \sqrt{x+36} = 3 + \sqrt{x} \)

\[ \iff (\sqrt{x+36})^2 = (3+\sqrt{x})^2 \]

\[ x + 36 = 9 + 6\sqrt{x} + x \]

\[ -x - 9 - 9 - x \]

\[ 27 = 6\sqrt{x} \]

\[ \iff \frac{27}{6} = \frac{9}{2} = \sqrt{x} \]

\[ \iff \left(\frac{9}{2}\right)^2 = (\sqrt{x})^2 \]

\[ \frac{81}{4} = x \]
#7b. \[ |x + 1| \leq 3 \]

\[ -3 \leq x + 1 \leq 3 \]

\[ -1 \quad -1 \quad -1 \]

\[ -4 \leq x \leq 2 \]
Intermediate Value Theorem (IVT)

Let $a$ and $b$ be real numbers such that $a < b$. If $f$ is a polynomial function such that $f(a) \neq f(b)$, then in the interval $[a,b]$, $f$ takes on every value between $f(a)$ and $f(b)$.

Example: Use the IVT to find intervals of length 1 in which $f$ is guaranteed to have a zero. Find the zeros. (Hint: Use the TABLE function on your calculator.)

11. $h(x) = x^4 - 10x^2 + 2$
\[ h(x) = x^4 - 10x^2 + 2 \]

- \([0, 1]\] in \([0, 1]\)
- \([3, 4]\] in \([3, 4]\)
- \([-4, -3]\]
- \([-1, 0]\] etc.
3.3 Real Zeros of Polynomial Functions

Long Division of Polynomials

Example: \[
\frac{6x^3 - 16x^2 + 17x - 6}{3x - 2} = 2x^2 - 4x + 3
\]

\[
\begin{align*}
3x-2 & \overline{6x^3-16x^2+17x-6} \\
\underline{- (6x^3-4x^2)} & \\
\quad & \underline{- (12x^2+17x)} \\
\qquad & \underline{- (-12x^2+8x)} \\
\quad & \underline{- (9x-6)} \\
\quad & 0
\end{align*}
\]
The Division Algorithm

If \( f(x) \) and \( d(x) \) are polynomials such that \( d(x) = 0 \), and the degree of \( d(x) \) is less than or equal to the degree of \( f(x) \), there exist unique polynomials such \( q(x) \) and \( r(x) \) such that

\[
f(x) = d(x)q(x) + r(x)
\]

\[
\text{dividend} = (\text{divisor})(\text{quotient}) + \text{remainder}
\]

where \( r(x) = 0 \) or the degree of \( r(x) \) is less than the degree of \( d(x) \). If the remainder \( r(x) \) is zero, \( d(x) \) divides evenly into \( f(x) \).

\[
\frac{f(x)}{d(x)} \text{ is improper, } \frac{r(x)}{d(x)} \text{ is proper.}
\]

For \[
\frac{6x^3 - 16x^2 + 17x - 6}{3x - 2}
\]

Write as:

\[
6x^3 - 16x^2 + 17x - 6 = (3x-2)(2x^2 - 4x + 3) + 0
\]

\[
\text{dividend} = (\text{divisor})(\text{quotient}) + \text{remainder}
\]

Because remainder = 0

\( 3x-2 \) divides evenly into \( 6x^3 - 16x^2 + 17x - 6 \)
Example: Perform long division and write in the form $f(x) = d(x)q(x) + r(x)$.

1. \[
\frac{x^5 + 7}{x^3 - 1} = x^2 + \frac{x^2 + 7}{x^3 - 1}
\]

\[
x^5 + 7 = (x^3 - 1)(x^2) + (x^2 + 7)
\]
Synthetic Division  (See pattern for dividing a cubic polynomial on page 267.)

Synthetic Division is a short-cut process for dividing a polynomial of any degree by a polynomial of the form \(x - k\).

Example: Use synthetic division to divide.

2. \[
\begin{array}{c|ccccc}
   \hline
   x+3 & 5x^3 + 18x^2 + 7x - 6 \\
   \hline
   -3 & 5 & 18 & 7 & -6 \\
   \hline
      3 & -15 & -9 & 6 \\
   \hline
      0 & 5 & 3 & -2 & 10
\end{array}
\]

Rightarrow \((x + 3)\) divides evenly into \(5x^3 + 18x^2 + 7x - 6\) and \((x + 3)\) is a factor of \(5x^3 + 18x^2 + 7x - 6\) and \(x = -3\) is a zero

Title: Oct 31 - 10:30 AM (8 of 19)
5x^3 + 18x^2 + 7x - 6 = (x+3)(5x^2 + 3x - 2)

On dividing by \( x+3 \), \( r = 0 \);
\[ f(-3) = \]

The Remainder Theorem
If a polynomial \( f(x) \) is divided by \( x - k \), the remainder is \( r = f(k) \).

The Factor Theorem
A polynomial \( f(x) \) has a factor \( (x - k) \) if and only if \( f(k) = 0 \).
Using the Remainder in Synthetic Division
In summary, the remainder $r$, obtained in synthetic division of $f(x)$ by $x - k$, provides the following information.

1. The remainder $r$ gives the exact value of $f$ at $x = k$. That is, $r = f(k)$.
2. If $r = 0$, $(x - k)$ is a factor of $f(x)$.
3. If $r = 0$, $(k, 0)$ is an $x$-intercept of the graph of $f$. 
Examples:

3. Use synthetic division and the Remainder Theorem to find $f(6)$ for $f(x) = 10x^4 - 50x^3 - 800$.

\[
\begin{array}{c|ccccc}
6 & 10 & -50 & 0 & 0 & -800 \\
\hline
 & 60 & 60 & 360 & 2160 \\
 & & 60 & 60 & 360 & 1360 \\
\end{array}
\]

\[\Rightarrow f(6) = 1360\]
4. Show that \( x = -2 \) is a zero (or root) of 
\( x^3 + 2x^2 - 2x - 4 \). Factor completely and find all real zeros.

\[
\begin{array}{c|cccc}
-2 & 1 & 2 & -2 & -4 \\
 & -2 & 0 & 4 \\
\hline
1 & 0 & -2 & 0 \Rightarrow -2 \text{ is a zero} \ (x+2) \text{ is a factor} \\
\end{array}
\]

\[
\text{other factor}
\]

\[
\Rightarrow x^3 + 2x^2 - 2x - 4 = (x + 2)(x^2 - 2)
\]

\[
= (x + 2)(x - \sqrt{2})(x + \sqrt{2})
\]

\[
\text{completely factored over the reals}
\]

Real zeros are: \( -2 \pm \sqrt{2} \)
5. Use the **Zero** feature of your calculator to approximate the zeros of \( f(s) = s^3 - 12s^2 + 40s - 24 \) to three decimal places. Determine one of the exact zeros and use synthetic division to verify it. Factor completely.

\[
\begin{array}{c|cccc}
-6 & 1 & -12 & 40 & -24 \\
 & & 6 & -36 & 24 \\
\hline
 & 1 & -6 & 4 & 0
\end{array}
\]

so \( 6 \) is a zero, \( (s-6) \) is a factor

\[
s^3 - 12s^2 + 40s - 24 = (s - 6)(s^2 - 6s + 4)
\]

use quad form.

\[
s = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

\[
= \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(4)}}{2(1)} = \frac{6 \pm \sqrt{120}}{2}
\]

\[
= \frac{6 \pm 2\sqrt{30}}{2} = 3 \pm \sqrt{30}
\]

in exact form

Title: Oct 31 - 10:31 AM (13 of 19)
\( f(s) = s^3 - 12s^2 + 40s - 24 \)

\[= \] (over rationals)

\[= (s-6)(s^2-6s+4) \]

(over reals)

\[= (s-6)(s-3-\sqrt{15})(s-3+\sqrt{15}) \]

If \( k \) is a zero, \( s-k \) is a factor.
The Rational Zero Test

If the polynomial $f(x) = a_n x^n + \ldots + a_1 x + a_0$ has integer coefficients, every rational zero of $f$ has the form

$$\text{Rational zero} = \frac{p}{q}$$

Where $p$ and $q$ have no common factors other than 1, $p$ is a factor of the constant term $a_0$ and $q$ is a factor of the leading coefficient $a_n$. 
Examples: List all possible rational zeros.

6. \( f(x) = 4x^4 - 17x^2 + 4 \)

\( g \) is a factor

\( p: \pm 1, \pm 2, \pm 4 \)

\( q: 1, 2, 4 \)

\( p \) is a factor

\( \frac{b}{q}: \frac{\pm 1}{\pm 2} = \pm \frac{1}{2} \)
7. \[ f(x) = 6x^3 - x^2 - 13x + 8 \]

- \( q \) is a factor
- \( p \) is a factor

\( p: \pm 1, \pm 2, \pm 4, \pm 8 \)

\( q: \ 1, 2, 3, 6 \)

\[ \frac{P}{q} : \pm 1, \pm 2, \pm 4, \pm 8, \pm \frac{1}{2}, \pm \frac{1}{3}, \pm \frac{2}{3}, \pm \frac{4}{3}, \pm \frac{8}{3}, \pm \frac{1}{6} \]
Skip Descartes’ Rule of Signs and Upper and Lower Bounds.

Examples: Find all real zeros.

8. \( h(x) = -x^3 - 9x^2 + 20x - 12 \)

\[ 
\begin{array}{c|cccc}
-12 & 1 & -9 & 20 & -12 \\
\hline 
& 12 & -36 & 192 & \hline
\end{array} 
\]

\( f(-12) = 180 \)

???
9. \( f(z) = 12z^3 - 4z^2 - 27z + 9 \)

From graph on calculator and ZERO command, 
\( z = 1.5 = \frac{3}{2} \) is a zero.

\[ \begin{array}{cccc}
2 & 3 & 12 & -4 & -27 & 9 \\
\hline
18 & 21 & -9 & 0 & 0 & 0 \\
\end{array} \]

\( 12, 14, -6 \parallel 0 \Rightarrow \frac{3}{2} \) is a zero

\( (z - \frac{3}{2}) \) is a factor

\[ 12z^2 + 14z - 6 = 2(6z^2 + 7z - 3)(z - \frac{3}{2}) \]

\[ = (2z - 3)(2z + 3)(3z - 1) \]

zeros: \( \frac{3}{2}, -\frac{3}{2}, \frac{1}{3} \)

using QUAD

Homework Section 3.3