Section 1.4 (Homework)

42. \( f(x) = \begin{cases} 
  x + 6 & x \leq -4 \\
  2x - 4 & x > -4 
\end{cases} \)

\[
\begin{array}{c|c|c}
 x+6 & x \leq -4 \\
 -4 & 2 \ \ (4, 2) \\
 -6 & 0 \ \ (-6, 0) \\
\end{array}
\]

\[
\begin{array}{c|c|c}
 2x - 4 & x > -4 \\
 -4 & -12 \ \ (-4, -12) \\
 0 & -4 \ \ (0, -4) \\
2 & 0 \ \ (2, 0) \\
\end{array}
\]
1.5 Shifting, Reflecting, and Stretching Graphs

Graphs of Common Functions

(a.) Constant Function  (b.) Identity Function  (c.) Absolute Value Function

(d.) Square Root Function  (e.) Quadratic Function  (f.) Cubic Function
Vertical and Horizontal Shifts: Let $c$ be a positive real number. Vertical and horizontal shifts of $y = f(x)$ are represented as follows:

1. Vertical shift $c$ units upward: $h(x) = f(x) + c$
2. Vertical shift $c$ units downward: $h(x) = f(x) - c$
3. Horizontal shift $c$ units to the right: $h(x) = f(x - c)$
4. Horizontal shift $c$ units to the left: $h(x) = f(x + c)$
Reflections in the Coordinate Axes: Reflections in the coordinate axes of the graph of $y = f(x)$ are represented as follows:

1. Reflection in the x-axis: $h(x) = -f(x)$
2. Reflection in the y-axis: $g(x) = f(-x)$
Examples: Sketch the graphs of the following functions starting with the graph of one of the six common functions.

\[ h(x) = -(x - 2)^2 + 4 \]

\[ f(x) = x^2 \]
\[ g(x) = (x-2)^2 \text{ [hor. shift 2 units right]} \]
\[ k(x) = -(x-2)^2 \text{ [reflection in x-axis]} \]
\[ h(x) = -(x-2)^2 + 4 \text{ [vert. shift 4 units up]} \]
$$w(x) = -x + 3$$

$$f(x) = x$$

$$g(x) = -x$$

$$w(x) = -x + 3$$
Example: Use the graph of \( f(x) = x^2 \) to write formulas for the functions \( g \) and \( h \) shown in the given graph.

1. \( f(x+1) = (x+1)^2 \)
2. \( g(x) = -(x+1)^2 \)

\[
g(x) = - (x+1)^2
\]

\[
h(x) = (x+2)^2 - 3
\]
Definitions: Horizontal and vertical shifts and reflections in the coordinate axes are called rigid transformations because the basic shape of the graph is unchanged – only the position is changed. Nonrigid transformations are those that cause a distortion - a change in the shape of the original graph.
Some examples of nonrigid transformations of the graph of the function \( y = f(x) \) are listed below.

1. Vertical Stretch \( h(x) = cf(x) \), where \( c > 1 \)
2. Vertical Shrink \( h(x) = cf(x) \), where \( 0 < c < 1 \)
3. Horizontal Stretch \( h(x) = f(cx) \), where \( 0 < c < 1 \)
4. Horizontal Shrink \( h(x) = f(cx) \), where \( c > 1 \)

Let \( h(x) = 4x^2 \) and \( h(x) = (2x)^2 \). Both vertical stretch by factor of 4 and horizontal shrink by a factor of 2.
\[ h(x) = \left( \frac{x}{2} \right)^2 = \frac{x^2}{4} \]

\[ h(x) = \frac{1}{4} x^2 \quad 0 < \frac{1}{4} < 1 \]

(Vertical shrink by factor of \( \frac{1}{4} \))

\[ h(x) = \left( \frac{x}{2} \right)^2 \quad c = \frac{1}{2} < 1 \]

(Horizontal stretch by factor of 2)
Example: List the transformations required to get from \( f(x) = x^3 \) to \( w(x) = -2x^3 + 1 \).

- **2 steps**
- \( f(x) = x^3 \)
- \( g(x) = 2x^3 \) (vert. stretch by factor of 20)
- \( h(x) = -2x^3 \) (reflection in x-axis)
- \( w(x) = -2x^3 + 1 \) (vert. shift 1 unit up)
1.6 Combinations of Functions

Sum, Difference, Product, and Quotient of Functions

Let $f$ and $g$ be two functions with overlapping domains. Then, for all $x$ common to both domains, the sum, difference, product, and quotient of $f$ and $g$ are defined as follows.

Sum: \[(f + g)(x) = f(x) + g(x)\]

Difference: \[(f - g)(x) = f(x) - g(x)\]

Product: \[(fg)(x) = f(x) \cdot g(x)\]

Quotient: \[\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, g(x) = 0\]
Example: Given \( f(x) = \frac{x}{x+1}, g(x) = x^3 \), find \( f + g, f - g, fg, \frac{f}{g} \).

1. \((f+g)(x) = \frac{x}{x+1} + x^3 = \frac{x+x^4+x^3}{x+1} \)

2. \((f-g)(x) = \frac{x}{x+1} - x^3 = \frac{x-x^4-x^3}{x+1} \)

3. \((fg)(x) = \left(\frac{x}{x+1}\right)x^3 = \frac{x^4}{x+1} \)

   Domain of \( f+g, f-g \) and \( fg \) is \( x \neq -1 \) (domain of \( f \))
\[ \frac{f}{g}(x) = \frac{x}{x+1} = \frac{x}{(x+1)x^2} \]

\[ = \frac{1}{x+1}x^2 \]

\[ \text{domain: } x \neq -1, 0 \]

\[ \left( \frac{g}{f} \right)(x) = \frac{x^3}{x(x+1)} = \frac{x^3}{x+1} \]

\[ = x^2(x+1) \]

\[ \text{domain: } x \neq -1, 0 \]
Example: Given \( f(x) = \frac{x}{2}, g(x) = \sqrt{x} \), sketch the graphs of \( f, g, \) and \( f + g \).

\[ \begin{align*}
\text{domain of } f & : \mathbb{R} \\
\text{domain of } g & : x \geq 0 \\
\text{domain of } f + g & : x \geq 0
\end{align*} \]

Domain of \( f + g \) is restricted to the overlap of the domains of \( f \) and \( g \).
Definition: The composition of the function \( f \) with the function \( g \) is \((f \circ g)(x) = f(g(x))\). The domain of \( f \circ g \) is the set of all \( x \) in the domain of \( g \) such that \( g(x) \) is in the domain of \( f \).

Examples: In each of the following, find 
\((f \circ g)(x), (g \circ f)(x), (f \circ f)(x)\). (Problems from p. 142)

36. \( f(x) = \sqrt[3]{x-1}, g(x) = x^3 + 1 \)  

\[
(f \circ g)(x) = f(g(x)) = \sqrt[3]{g(x)} - 1 = \sqrt[3]{(x^3 + 1)} - 1 = \sqrt[3]{x^3} = x
\]

\[
(g \circ f)(x) = g(f(x)) = [f(x)]^3 + 1 = [\sqrt[3]{x-1}]^3 + 1 = x-1 + 1 = x
\]

\[
(f \circ f)(x) = f(f(x)) = \sqrt[3]{f(x)-1}
\]
38. \( f(x) = x^3, \quad g(x) = \frac{1}{x}, \quad x \neq 0 \)

\[
(f \circ g)(x) = f(g(x)) = \left[ g(x) \right]^3 = \left( \frac{1}{x} \right)^3 = \frac{1}{x^3}
\]

\[
(g \circ f)(x) = g(f(x)) = \frac{1}{f(x)} = \frac{1}{x^3}
\]

\[
(f \circ f)(x) = f(f(x)) = \left[ f(x) \right]^3 = \left[ x^3 \right]^3 = x^9
\]
52. Use the graphs of \( f \) and \( g \) to evaluate (a.) \((f-g)(1)\) and (b.) \((fg)(4)\).

\[
\begin{align*}
a.) \quad (f-g)(1) &= f(1) - g(1) \\
&= 2 - 3 = -1
\end{align*}
\]

\[
\begin{align*}
b.) \quad (fg)(4) &= (f(4))(g(4)) \\
&= (4)(0) = 0
\end{align*}
\]
56. Find two functions \( f \) and \( g \) such that 
\[
(f \circ g)(x) = h(x) \text{ if } h(x) = (1-x)^3.
\]

\[
(f \circ g)(x) = f(g(x))
\]

\[
g(x) = 1-x
\]

\[
f(x) = x^3
\]

\[
f(g(x)) = (g(x))^3 = (1-x)^3 = h(x)
\]

*note: Let \( g(x) = \frac{1-x}{2} \)

\[
f(x) = 8x^3
\]

\[
(f \circ g)(x) = f(g(x)) = 8[g(x)]^3 = 8(\frac{1-x}{2})^3
\]

\[
= 8(1-x)^3 = (1-x)^3 = h(x)
\]
Homework Sections 1.5 and 1.6

Test Tuesday, Oct. 11.