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1.1 Introduction to Systems of Linear Equations

Linear equation in $x$ and $y$: $a_1x + a_2y = b$

Linear equation in $n$ variables: $a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$

$a_1, a_2, \ldots, a_n, b$ are real constants

The solution of the equation $a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$ is a sequence of $n$ numbers $s_1, s_2, \ldots, s_n$ such that the equation is satisfied when we substitute $x_1 = s_1$, etc.

The set of all solutions of the equation is called its solution set or sometimes the general solution.
Examples:

1. Find the solution set for the equation: \(3x - 4y = 8\)

   \[
   3x - 8 = 4y
   \]

   Let \(x = t\).

   \[
   y = \frac{3}{4}x - 2
   \]

   \[
   \begin{cases} 
   x = t \\
   y = \frac{3}{4}t - 2 
   \end{cases}
   \]

   A particular solution, for example, would be \(x = 4, y = 1\).

   Let \(t = 0\) \(x = 0, y = -2\)

   \(t = 2\) \(x = 2, y = -\frac{1}{2}\) etc.
2. Find the solution set of the equation

\[-3x_1 + 4x_2 - 7x_3 + 8x_4 = 5\]

Let \(x_2 = s, x_3 = t, x_4 = u\)

\[
x_1 = \frac{-4s + 7t - 8u + 5}{-3} = \frac{4}{3}s - \frac{7}{3}t + \frac{8}{3}u + 5
\]

\[
\begin{align*}
    x_1 &= \frac{4}{3}s - \frac{7}{3}t + \frac{8}{3}u + 5 \\
    x_2 &= s \\
    x_3 &= t \\
    x_4 &= u
\end{align*}
\]

general solution

particular solution

Let \(s = t = u = 0\), \(x_1 = 5\), \(x_2 = 0\), \(x_3 = 0\), \(x_4 = 0\)

Let \(s = 0\), \(x_1 = \frac{4}{3}(0) - \frac{7}{3}(3) + \frac{8}{3}(-3) + 5\)
\(\begin{align*}
    s &= 3 \\
    t &= 0 \\
    u &= -3
\end{align*}\)

\(x_1 = -10\), \(x_2 = 0\), \(x_3 = 3\), \(x_4 = -3\)
A finite set of linear equations in the variables $x_1, x_2, \ldots, x_n$ is called a **system of linear equations** or a **linear system**.

A **solution** of the system is a sequence of $n$ numbers $s_1, s_2, \ldots, s_n$ such that $x_1 = s_1, x_2 = s_2, \ldots$, etc. is a solution of every equation in the system.

Not all systems have solutions. If a system has no solution it is said to be **inconsistent**. If a system has at least one solution, it is said to be **consistent**.
What is true for systems of 2 linear equations in 2 variables is true for linear systems of any size: there will always be 0, 1, or an infinite number of solutions.
Notation: A system of \( m \) linear equations in \( n \) variables may be written as:

\[
\begin{align*}
    a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
    a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\
    &\vdots \\
    a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \\
\end{align*}
\]

\( a_{ij} = \) entry in the \( i \)th equation that multiplies \( x_j \).

The augmented matrix for this system is:

\[
\begin{bmatrix}
    a_{11} & \ldots & a_{1n} & b_1 \\
    a_{21} & \ldots & a_{2n} & b_2 \\
    \vdots & \ddots & \vdots & \vdots \\
    a_{m1} & \ldots & a_{mn} & b_m \\
\end{bmatrix}
\]

Size: \( m \times (n+1) \) (rows \times columns)
Example 3: Find the augmented matrix for the system

\[
\begin{align*}
3x_1 - 4x_2 - 2x_3 &= 5 \\
x_1 - 5x_2 &= 3 \\
-2x_1 - x_2 + 3x_3 &= -2 \\
&\ldots
\end{align*}
\]

\[
\begin{bmatrix}
3 & -4 & -2 & 5 \\
1 & -5 & 0 & 3 \\
-2 & -1 & 3 & -2
\end{bmatrix}
\]
Method for Solving a System of Linear Equations - Replace the given system with a new system that has the same solution but is easier to solve, using the following operations:

1. Multiply an equation through by a nonzero constant.
2. Interchange two equations.
3. Add a multiple of one equation to another equation.

Since the rows of the augmented matrix correspond to the equations in the associated system, there are three elementary row operations that correspond to the above operations:

1. Multiply a row through by a nonzero constant.
2. Interchange two rows.
3. Add a multiple of one row to another row.
Example 4: Solve using elementary row operations.

\[3x - 4y = -10\]
\[-5x + 3y = 2\]

\[
\begin{bmatrix}
3 & -4 & -10 \\
-5 & 3 & 2
\end{bmatrix}
\begin{bmatrix}
1 \\
\frac{1}{3}
\end{bmatrix}
= R_1
\begin{bmatrix}
1 & -\frac{4}{3} & -\frac{10}{3} \\
-5 & 3 & 2
\end{bmatrix}
\]

\[
5R_1 + R_2 = R_2
\begin{bmatrix}
1 & -\frac{4}{3} & -\frac{10}{3} \\
0 & -\frac{11}{3} & -\frac{44}{3}
\end{bmatrix}
\begin{bmatrix}
-\frac{3}{11} \\
R_2 = R_2
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & -\frac{4}{3} & -\frac{10}{3} \\
0 & 1 & 4
\end{bmatrix}
\Rightarrow y = 4; x = 2
\]

\[
\frac{1}{4} \cdot 4 = \frac{10}{3}
\]

\[x = \frac{10}{3} \quad \text{let } y = 4
\]

\[
x = \frac{4}{3} (4) = -\frac{10}{3}
\]

\[
x = \frac{10}{3} \quad x = \frac{6}{3} = 2
\]
1.2 Gaussian Elimination

To be in **reduced row-echelon form**, a matrix must have the following properties:

1. If the row does not consist entirely of zeros, then the first nonzero number in the row is a 1. We call this a **leading 1**.

2. If there are any rows that consist entirely of zeros, then they are grouped together at the bottom of the matrix.

3. In any two successive rows that do not consist entirely of zeros, the leading 1 in the lower row occurs farther to the right than the leading 1 in the higher row.

4. Each column that contains a leading 1 has zeros everywhere else.
Example: (both are in reduced row-echelon form)

\[
\begin{bmatrix}
1 & 0 & 0 & 2 \\
0 & 1 & 0 & -3 \\
0 & 0 & 1 & 5 \\
\end{bmatrix},
\begin{bmatrix}
0 & 1 & 3 & 0 & 2 \\
0 & 0 & 0 & 1 & -3 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

If a matrix has the first 3 properties but not the 4\textsuperscript{th}, it is said to be in row-echelon form.

Example:

\[
\begin{bmatrix}
1 & 2 & 3 & 5 \\
0 & 1 & 2 & 4 \\
0 & 0 & 1 & 0 \\
\end{bmatrix}
\]

(row-echelon form

(not reduced row-echelon form)
If we assume that the first matrix,
\[
\begin{bmatrix}
1 & 0 & 0 & 2 \\
0 & 1 & 0 & -3 \\
0 & 0 & 1 & 5
\end{bmatrix}
\]
represents the reduced row-echelon form of the augmented matrix for a system of linear equations, we may read off the solution to the system:

\[
\begin{align*}
x_1 &= 2 \\
x_2 &= -3 \\
x_3 &= 5
\end{align*}
\]

\(x_1, x_2, x_3\) are called leading variables.
Examples 1: Find the solution to the system of linear equations whose reduced augmented matrix is given.

\[
\begin{bmatrix}
0 & 1 & 3 & 0 & 2 \\
0 & 0 & 0 & 1 & -3 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[x_2 + 3x_3 = 2\]
\[x_4 = -3\]

Let \(x_1 = u, x_3 = t\). Then \(x_1 = u\)

just like in 2-space \(y = 2\) (horizontal line)
\(x\) can take on any value

The leading variables are \(x_2, x_4\)
Suppose one row of the matrix is
\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]
\[0x_1 + 0x_2 + 0x_3 + 0x_4 = 1\]
\[0 = 1\quad \text{false}\]
This equation has no solution implying that the system has no solution.

**Gauss-Jordan elimination** is the procedure used to reduce any matrix to reduced row-echelon form.
Example 2: Solve the given linear system by Gauss-Jordan elimination.

\[
\begin{align*}
    x_1 - 2x_2 + x_3 - 4x_4 &= 1 \\
    x_1 + 3x_2 + 7x_3 + 2x_4 &= 2 \\
    x_1 - 12x_2 - 11x_3 - 16x_4 &= 5
\end{align*}
\]

\[
\begin{bmatrix}
    1 & -2 & 1 & -4 & 1 \\
    1 & 3 & 7 & 2 & 2 \\
    1 & -12 & -11 & -16 & 5
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
    1 & -2 & 1 & -4 & 1 \\
    0 & 1 & \frac{6}{5} & \frac{6}{5} & \frac{1}{5} \\
    0 & 0 & 0 & 0 & 6
\end{bmatrix}
\]

Therefore, the system has no solution.
If we use the same procedures to reduce the matrix to row-echelon form, we call it **Gaussian elimination**. To solve a matrix in row-echelon form it is usually necessary to use a procedure called **back substitution**.

**Example 3:** The given matrix is in row-echelon form. Use back substitution to solve the system.

\[
\begin{bmatrix}
1 & -2 & 1 & 1 \\
0 & 1 & 2 & 2 \\
0 & 0 & 1 & 3 \\
\end{bmatrix}
\]

\[x_1 - 2x_2 + x_3 = 1\]
\[x_2 + 2x_3 = 2\]
\[x_2 = 3\]

\[x_1 + 8 + 3 = 1\quad (x_1 = -10)\]
\[x_1 - 2(-4) + 3 = 1\quad (x_1 = -10)\]
\[x_2 + 2(3) = 2\quad (x_2 = -4)\]
\[x_2 + 6 = 2\quad (x_2 = -4)\]

\[\Rightarrow (x_1 = -10, x_2 = -4, x_3 = 3)\]
A system of linear equations is said to be **homogeneous** if all of the constant terms are zero.

\[ a_{11}x_1 + ... + a_{1n}x_n = 0 \]

\[ a_{21}x_1 + ... + a_{2n}x_n = 0 \]

\[ ... \]

\[ a_{31}x_1 + ... + a_{3n}x_n = 0 \]

Every homogeneous system is consistent because it has the solution \( x_1 = x_2 = ... = x_n = 0 \), the **trivial solution**. Any other solutions are **nontrivial solutions**.
Exactly one of the following is true:

1. The system has only the trivial solution.
2. The system has infinitely many nontrivial solutions in addition to the trivial solution.

The augmented matrix for a homogeneous system has these properties:

1. No elementary row operation can alter a column of zeros.
2. The number of equations in the reduced system is the same or less than the number in the original system.
Theorem 1.2.1 A homogeneous system of linear equations with more unknowns than equations has infinitely many solutions.

(This is true for nonhomogeneous systems only if they are consistent.)
Example 4: Find all solutions to the homogeneous system.

\[ \begin{align*}
2x_1 - 4x_2 + x_3 + x_4 &= 0 \\
-x_1 - 5x_2 + 2x_3 &= 0 \\
-2x_2 - 2x_3 - x_4 &= 0 \\
x_1 + 3x_2 + x_4 &= 0 \\
x_1 - 2x_2 - x_3 + x_4 &= 0
\end{align*} \]

\[
x_1 = x_2 = x_3 = x_4 = 0
\]
Example 5: (From text, page 21, #22)

For which value(s) of $\lambda$ does the system of equations

1. $(\lambda - 3)x + y = 0$
2. $x + (\lambda - 3)y = 0$

have nontrivial solutions?

(Substitution method)

$\Rightarrow \begin{align*}
\text{Substitute in } \text{eq } 2 & \Rightarrow y = -(\lambda - 3)x \\
\text{Substitute in eq } 2 & \Rightarrow x + (\lambda - 3)[-(\lambda - 3)x] = 0 \\
\Rightarrow x & - (\lambda - 3)^2 x = 0 \\
x & \left[1 - (\lambda^2 - 6\lambda + 9)\right] = 0 \\
x & \left[-\lambda^2 + 6\lambda - 8\right] = 0
\end{align*}$

1. if $x = 0$, $y = 0$ trivial soln.
2. if $x \neq 0$, then $-\lambda^2 + 6\lambda - 8 = 0$ (non-trivial) or $\lambda^2 - 6\lambda + 8 = 0$ (non-trivial)

$\lambda = 2, 4$

Homework: Sections 1.1, 1.2