8.1, 8.2, 8.3, 8.6  Solving Systems of Equations

8.1 The Method of Substitution (See steps page 592)

Examples: Solve by the Method of Substitution. Check your answer by using a graphing utility.

1. \[ x + 2y = 1 \] => \[ x = 1 - 2y \]

\[ 5x - 4y = -23 \]

\[
\begin{align*}
5(1-2y) - 4y &= -23 \\
5 - 10y - 4y &= -23 \\
-5 &= -14y \\
-5 &= -14y \\
\frac{-14y}{-14} &= 3 \\
\frac{-14y}{14} &= 3 \\
\end{align*}
\]

\[ y = 3 \]

\[ x = 1 - 2y = 1 - 2(3) = -5 \]

\[ (x, y) = (-5, 3) \]
2. \[ x^2 + y^2 = 25 \]
\[ 2x + y = 10 \Rightarrow y = 10 - 2x \]

\[
\begin{align*}
\star & \quad x^2 + y^2 = 25 \\
& \quad x^2 + (10 - 2x)^2 = 25 \\
& \quad x^2 + 100 - 40x + 4x^2 = 25 \\
& \quad 5x^2 - 40x + 75 = 0 \\
& \quad 5(x^2 - 8x + 15) = 0 \\
& \quad 5(x - 3)(x - 5) = 0 \quad x = 3, 5 \\
\end{align*}
\]

\[
\begin{align*}
\star & \quad y = 10 - 2x \\
& \quad x = 3 \Rightarrow y = 10 - 2(3) = 4 \\
& \quad (x, y) = (3, 4) \\
& \quad x = 5 \Rightarrow y = 10 - 2(5) = 0 \\
& \quad (x, y) = (5, 0) \\
\end{align*}
\]

(calculator - "will not" find the 2nd point)
8.2 The Method of Elimination (See steps page 604.)

Definitions:
A system of equations is consistent if it has at least one solution. (For a linear system, one or an infinite number of solutions)
A system of equations is inconsistent if it has no solution.

Examples: Solve by using the Method of Elimination.

\[
\begin{align*}
2r + 4s &= 5 \\
16r + 50s &= 55
\end{align*}
\]

\[
\begin{align*}
\text{mult. by } 8 & \quad 16r - 32s = -40 \\
\text{mult. by } 25 & \quad 16r + 50s = 55
\end{align*}
\]

\[
18s = 15 \\
s = \frac{5}{6}
\]

\[
18r = 15 \\
r = \frac{5}{6}
\]

\[
(r, s) = \left(\frac{5}{6}, \frac{5}{6}\right)
\]
4. \[7x - 6y = -6\]
\[-7x + 6y = -4\]
\[0 = -10\]
Clearly false
\[\Rightarrow \text{no solution}\]

Note: \[-6y = -7x - 6 \Rightarrow y = \frac{-7}{6}x + 1\]
\[by = 7x - 4\]
\[\Rightarrow \text{parallel}\]
8.3 Multivariate Linear Systems

Definition: A system of linear equations that contains more than two variables is called a multivariate linear system.

Two systems of equations are equivalent if they have the same solution set.

A simple form of a system of linear equations is called row-echelon form. (See an example on page 613.)
A method used to reduce a system to row-echelon form is called Gaussian elimination and uses the three elementary row operations:

1. Interchange two equations.
2. Multiply one of the equations by a nonzero constant.
3. Add a multiple of one equation to another equation.
Examples: Solve by Gaussian elimination.

3. \[ 3x - 2y - 6z = -4 \]
5. \[ -3x + 2y + 6z = 1 \] (Inconsistent)
\[ x - y - 3z = -3 \]

\[ \begin{align*}
3x - 2y - 6z &= -4 \\
-3x + 2y + 6z &= 1 \\
\hline
0 &= -3 \text{ clearly false} \\
\Rightarrow &\text{ inconsistent}
\end{align*} \]

**augmented matrix**
\[ \begin{bmatrix}
3 & -2 & -6 & 4 \\
1 & -1 & -5 & -3
\end{bmatrix} \]

3 rows, 4 columns
Size: 3 \times 4

x + 4z = 0
y + 9z = 0
0 = 1
\Rightarrow \text{ inconsistent}
**

6. \( x - 3y + 2z = 18 \) (multiply by -5)

\[
\begin{align*}
5x - 13y + 12z &= 80 \\
-5x + 15y - 10z &= -90
\end{align*}
\]

\[
\begin{align*}
2y + 2z &= 10 \\
y + z &= 5 \quad \text{(y = -2 - 5)}
\end{align*}
\]

**

\[
\begin{align*}
x - 3y + 2z &= 18 \\
x - 3(-2 - 5) + 2z &= 18 \\
x + 3z + 15 + 2z &= 18 \\
x + 5z &= 3 \quad \text{(x = -5z + 3)}
\end{align*}
\]

Let \( z = t \)

\[
\begin{align*}
y &= -t - 5 \\
x &= -5t + 3
\end{align*}
\]

parametric solution

\[
\begin{bmatrix}
1 & 1 & -3 & 2 & 18 \\
15 & -13 & 12 & 80
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 0 & 5 & 3 \\
0 & 1 & 1 & -5
\end{bmatrix}
\]

\( x + 5z = 3 \)

\( y + z = -5 \)
\[2x + y + 3z = 1 \quad \text{(1)}\]
\[7. \quad 2x + 6y + 8z = 3 \quad \text{(2)}\]
\[6x + 8y + 18z = 5 \quad \text{(3)}\]

\[\text{Solution steps:}\]
\begin{align*}
2x + y + 3z &= 1 \\
-2x - 6y - 8z &= -3 \\
-6x - 9z &= -2 \\
-6x - 8y - 9z &= -3 \\
6x + 8y + 18z &= 5 \\
\end{align*}

\[5y + 9z = 2 \quad \text{(5)}\]

\[5y - 5z = -2 \\
\begin{align*}
5y + 9z &= 2 \\
4z &= 0 \\
z &= 0
\end{align*}

\[y = 2/5 \quad \text{(6)}\]

\[x = \frac{3}{10} \quad \text{(7)}\]

\[(x, y, z) = (\frac{3}{10}, \frac{2}{5}, 0)\]

*Let \(z = 0\), \(5y + 9(0) = 2\) → \(y = 2/5\)*

*Let \(y = 2/5\), \(2x + y + 3z = 1\) → \(z = 0\)*

*Let \(z = 0\), \(2x + \frac{2}{5} + 3(0) = 1\) → \(2x = \frac{3}{5} \Rightarrow x = \frac{3}{10}\)*
OR

\[
\begin{bmatrix}
2 & 1 & 3 & 1 \\
2 & 6 & 8 & 3 \\
6 & 8 & 18 & 5
\end{bmatrix}
\]

\[
\begin{bmatrix}
2 & 1 & 3 & 1 \\
2 & 6 & 8 & 3 \\
6 & 8 & 18 & 5
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 0 & 0 & 3/10 \\
0 & 1 & 0 & 3/5 \\
0 & 0 & 1 & 0
\end{bmatrix}
\]

\[
x = \frac{3}{10} \\
y = \frac{3}{5} \\
z = 0
\]