Section 1.5 (Homework)

#1. \[ f(x) = x \]
\[ g(x) = x - 4 \quad \frac{x^{10}}{g(x)^{2}} \]
\[ h(x) = 3x \]

\[
\begin{array}{c|c|c}
 x & 0 & 1 \\
 h(x) & 0 & 3 \\
\end{array}
\]
1.7 Inverse Functions

Definition: Let $f$ and $g$ be two functions such that
\[ f(g(x)) = x \] for every $x$ in the domain of $g$, and
\[ g(f(x)) = x \] for every $x$ in the domain of $f$.
Under these conditions, the function $g$ is the inverse function of the function $f$.
The function $g$ is denoted $f^{-1}$ (read “$f$-inverse”).

Therefore
\[ f(f^{-1}(x)) = x \text{ and } f^{-1}(f(x)) = x. \]
The domain of $f$ must be equal to the range of $f^{-1}$ and the range of $f$ must be equal to the domain of $f^{-1}$.
In a previous section, we looked at the functions:

\[ f(x) = \sqrt[3]{x - 1} \quad \text{and} \quad g(x) = x^3 + 1 \]

and found that

\[ f(g(x)) = x, \quad g(f(x)) = x \]

implying that \( f \) and \( g \) are inverses. The following shows the graph of \( y = f(x) \), \( y = g(x) \), and \( y = x \).

(Note that this is a Square screen.)

The graphs of \( f \) and \( g \) are symmetric in the line \( y = x \). This is true for any pair of inverse functions.
Examples: Using common sense, find the inverses of the following functions:

\[ f(x) = x + 2 \quad \Rightarrow \quad f^{-1}(x) = x - 2 \quad \Rightarrow \quad f(f^{-1}(5)) = f(3) = 5 \quad \checkmark \]

\[ f(x) = \frac{x}{3} \quad \Rightarrow \quad f^{-1}(x) = 3x \quad \Rightarrow \quad f(f^{-1}(5)) = f(15) = 5 \quad \checkmark \]

\[ g(x) = x^2 \quad \Rightarrow \quad g^{-1}(x) = \sqrt{x} \quad \Rightarrow \quad g(g^{-1}(9)) = g(3) = 9 \quad \checkmark \]

\[ x \geq 0 \]
Definition: A function \( f \) is one-to-one if, for \( a \) and \( b \) in its domain, \( f(a) = f(b) \) implies that \( a = b \).

A function \( f \) has an inverse function \( f^{-1} \) if and only if \( f \) is one-to-one.

\[
\text{examples: } \quad g(x) = x^2 \\
\text{but } \quad g(3) = g(-3) = 9 \\
\Rightarrow \quad g \text{ is not one-to-one}
\]

If we restrict the domain of \( g \) to be \( g(x) = x^2 \), \( x \geq 0 \), it is now one-to-one.
Horizontal Line Test: If every horizontal line intersects the graph of a function $f$ at most once, then the function is one-to-one. (That is, no horizontal line intersects the graph of the function more than once.)

Examples:

- One-to-one function
- Function, not one-to-one
- Not a function

(Don't even consider one-to-oneness)
Finding an inverse function:

1. Use the Horizontal Line Test to decide whether $f$ has an inverse function.

2. In the equation for $f(x)$, replace $f(x)$ with $y$.

3. Interchange the roles of $x$ and $y$, and solve for $y$.

4. Replace $y$ by $f^{-1}$ in the new equation.

5. Verify that $f$ and $f^{-1}$ are inverse functions by showing that the domain of $f$ is equal to the range of $f^{-1}$ and the range of $f$ is equal to the domain of $f^{-1}$ and that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$. 
Examples: In each case, determine if the given function is one-to-one. If so, find its inverse. (From p. 153)

44. \( f(x) = 3x + 5 \)

\[ \text{domain } \mathbb{R} \]
\[ \text{range } \mathbb{R} \]

\( f \) is one-to-one function

\( \text{Write } y = 3x + 5 \)

\( \text{Interchange } x \text{ and } y \)

\[ x = 3y + 5 \]

\( \text{Solve for } y: \)

\[ x - 5 = 3y \]

\[ y = \frac{x - 5}{3} \]

The new \( y \) is \( f^{-1} \):

\[ f^{-1}(x) = \frac{x - 5}{3} \]

\[ f^{-1}(x) = \frac{1}{3}x - \frac{5}{3} \]
Check:

\[ f(f^{-1}(x)) = \]

\[ 3f^{-1}(x) + 5 = \]

\[ 3 \left[ \frac{x-5}{3} \right] + 5 = \frac{x-5+5}{3} = x \]

Domain \( R \)

Range \( R \)
46. \[ h(x) = \frac{4}{x^2} \]

Fails horizontal line test \( \Rightarrow \)

it is not one-to-one \( \Rightarrow \)

it has no inverse
48. \( q(x) = (x - 5)^2, x \leq 5 \)  
\( \text{domain } x \leq 5 \)  
\( \text{range } y \geq 0 \)

\( q(x) \) satisfies horizontal line test.

\( \Rightarrow q(x) \) is one-to-one and has an inverse.

\( \therefore \) Write \( y = (x - 5)^2 \)

\( \therefore \) Interchange \( x \) and \( y \)

\( x = (y - 5)^2 \)

\( \therefore \) Solve for \( y \):

\( -\sqrt{x} = y - 5 \) or

\( \sqrt{x} = y - 5 \)

\( \Rightarrow y = \sqrt{x} + 5 \)

\( \Rightarrow q^{-1}(x) = \sqrt{x} + 5 \)
5. \( q(x) = (x - 5)^2 \), \( x \leq 5 \)

\( q^{-1}(x) = -\sqrt{x + 5} \)

\( x \geq 0 \)

Domain: \( y = 5 \)

\( q(q^{-1}(x)) = (q^{-1}(x) - 5)^2 \)

\( = (-\sqrt{x + 5} - 5)^2 \)

\( = (-\sqrt{x})^2 = x \) ✓
\[ f(x) = \sqrt{x-2} \]

**Domain:** \( x \geq 2 \)

**Range:** \( y \geq 0 \)

1. Satisfies horizontal line test
2. \( f \) is one-to-one and has an inverse

**Step 2:** Write \( y = \sqrt{x-2} \)

**Step 3:** Interchange \( x \) and \( y \):\[ x = \sqrt{y-2} \]

**Step 4:** Solve for \( y \): \[ x^2 + 2 = y \]

\[ f^{-1}(x) = x^2 + 2 \]

**Domain:** \( x \geq 0 \)

**Range:** \( y \geq 2 \)
\[ f(x) = \sqrt{x-2} \]
\[ f^{-1}(x) = x^2 + 2, \quad x \geq 0. \]

\[ f(f^{-1}(x)) = \sqrt{f^{-1}(x)} - 2 \]
\[ = \sqrt{x^2 + 2} - 2 = \sqrt{x^2} = x \]

\[ f^{-1}(f(x)) = (f(x))^2 + 2 = (\sqrt{x-2})^2 + 2 \]
\[ = x - 2 + 2 = x \]
2.1 Linear Equations and Problem Solving

Definitions:
An equation in \( x \) is a statement that two algebraic expressions are equal.
To solve an equation in \( x \) means to find all values of \( x \) for which the equation is true.
Values of \( x \) for which an equation is true are called its solutions.
An equation that is true for every real number in the domain of the variable is called an identity.
An equation that is true for just some (or even none) of the real numbers in the domain of the variable is called a conditional equation.
An equation which is not true for any real number is called a contradiction.
A linear equation in one variable $x$ is an equation that can be written in the standard form $ax + b = 0$ where $a$ and $b$ are real numbers, with $a \neq 0$. An extraneous solution is one that does not satisfy the original equation. It is often introduced when an equation is multiplied or divided by a variable expression.
Examples: Solve the following equations. Use a graphing utility to verify your solutions.

1. \[ \frac{x}{5} - \frac{x}{2} = 3 \]

\[
\frac{x}{5} - \frac{x}{2} = 3 \quad \text{(mult. by 10)}
\]
\[
20 \left( \frac{x}{5} \right) - 10 \left( \frac{x}{2} \right) = 10 \cdot 3
\]
\[
2x - 5x = 30
\]
\[
-3x = 30
\]
\[
\frac{-3x}{-3} = \frac{30}{-3}
\]
\[
x = -10
\]
2. \[
\frac{17 + y}{y} + \frac{32 + y}{y} = 100
\]

\[
y \left( \frac{17 + y}{y} \right) + y \left( \frac{32 + y}{y} \right) = y \left( 100 \right)
\]

\[
17 + y + 32 + y = 100 y
\]

\[
49 + 2y = 100 y
\]

\[
-2y = -2y
\]

\[
\frac{49}{98} = 0.5
\]

\[
\frac{98y}{98} = y
\]

\[
\frac{1}{2} = y
\]
3. \[
\frac{x}{x+4} + \frac{4}{x+4} + 2 = 0
\]
if \(x = -4\) (mult. by \(x+4\))

\[
(x+4) \left( \frac{x}{x+4} \right) + (x+4) \left( \frac{4}{x+4} \right) + (x+4)(2)
= (x+4)(10)
\]

\[
x + 4 + 2x + 8 = 0
\]

\[
3x + 12 = 0 \quad 3x = -12 \quad x = -4
\]

**No solution**

**Extraneous solution**

Left side is 3 except there is a hole in the graph at \(x = -4\)
Examples:

\[ 2(x+3) = 2x + 6 \]

True for all real numbers

\[ \implies \text{it is an identity} \]

\[ 2(x+3) = 2x + 4 \]

\[ 2x + 6 = 2x + 4 \]

\[ -2x - 6 = -2x - 6 \]

\[ 0 = -2 \] clearly false

no solution;

original equation is a contradiction.
Solve: \( \frac{5x}{4} + \frac{1}{2} = x - \frac{1}{2} \) (multiplied by 4)

\[ 4 \left( \frac{5x}{4} \right) + 4 \left( \frac{1}{2} \right) = 4x - 4 \left( \frac{1}{2} \right) \]

\[ 5x + 2 = 4x - 2 \]

\[ -4x + 2 = -4x - 2 \]

\[ x = -4 \]