Homework (Section P.2)

#19 b. \( \frac{a^{-2}}{b^{-2}} \left( \frac{b}{a} \right)^3 = \frac{b^5}{a^5} \)

#55 b. \( \sqrt[5]{96 \times 57} = \sqrt[5]{32 \times 5 \times 3} = 2 \times \sqrt[5]{3} \)

#37 b. \( \sqrt[3]{6.3 \times 10^4} \approx 39.791 \)
P.3 Polynomial and Factoring

Definitions:
Let \( a_0, a_1, a_2, \ldots, a_n \) be real numbers and let \( n \) be a nonnegative integer. A polynomial in \( x \) is an expression of the form
\[
a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0, a_n \neq 0.
\]
The polynomial is of degree \( n \), \( a_n \) is the leading coefficient, and \( a_0 \) is the constant term.
Other terms:

Standard form

Monomial (one term)
example: $3x^2$ or $-2$

Binomial (two terms)
example: $x - 2y$

Trinomial (three terms)
example: $x^2 - 5x + 6$

Zero polynomial $0$ (terms are in decreasing order)
Examples: (Operations with polynomials)

Perform the operations and write the result in standard form

\[-(5x^2 - 1) - (-3x^2 + 5) = -5x^2 + 1 + 3x^2 - 5\]

\[= -2x^2 - 4\]

\[-4x(3 - x^3) = -12x + 4x^4\]

\[= 4x^4 - 12x\]
\[(15.6x^4 - 18x - 19.4) - (13.9x^4 - 9.2x + 15)\]
\[= 15.6x^4 - 18x - 19.4 - 13.9x^4 + 9.2x - 15\]
\[= (1.7x^2 - 8.8x - 34.4)\]

\[(7x - 2)(4x - 3)\]
\[= 28x^2 - 21x - 8x + 6\]
\[= 28x^2 - 29x + 6\]
Special Products

1. Sum and Difference of Two Terms

\[(u - v)(u + v) = u^2 - v^2\]
\[(u - v)(u + v) = u^2 + uv - uv - v^2 = u^2 - v^2\]

2. Square of a Binomial

\[(u + v)^2 = u^2 + 2uv + v^2\]
\[(u - v)^2 = u^2 - 2uv + v^2\]

3. Cube of a Binomial

\[(u + v)^3 = u^3 + 3u^2v + 3uv^2 + v^3\]
\[(u - v)^3 = u^3 - 3u^2v + 3uv^2 - v^3\]

4. Sum and Difference of Cubes

\[u^3 + v^3 = (u + v)(u^2 - uv + v^2)\]
\[u^3 - v^3 = (u - v)(u^2 + uv + v^2)\]

**Proofs:**

\[(u + v)^3 = (u + v)(u + v)(u + v)\]
\[= (u^2 + 2uv + v^2)(u + v)\]
\[= u^3 + 2u^2v + uv + 2uv^2 + 2u^2v + uv + v^3\]
\[= u^3 + 3u^2v + 3uv^2 + v^3\]

\[u^3 - v^3 = (u - v)(u^2 + uv + v^2)\]
\[= (u^3 + u^2v + uv^2 + uv^2 + u^2v + uv^2 + v^3)\]
\[= u^3 - 3u^2v - 3uv^2 - v^3\]
Examples: Find the special product:

\[(x-2)^3\] use: \((u-v)^3 = u^3 - 3uv^2 + 3uv - v^3\]

Let \(u = x\)
\(v = 2\)

\[\begin{align*}
(x-2)^3 &= x^3 - 3x(2)^2 + 3x(2) - 2 \\
&= x^3 - 12x + 8
\end{align*}\]

\[\left[(x+1)^2 - 1\right]^2\] use \((u-v)^2 = u^2 - 2uv + v^2\)

Let \(u = (x+1)^2\)
\(v = 1\)

\[\left[(x+1)^2 - 1\right]^2 = \left[(x+1)^2 - 2(x+1)(1) + 1\right] \nonumber\]

\(\Rightarrow\) \(x^2 + 2x + 1\)

\[\begin{align*}
(x^2 + 2x + 1)^2 &= (x^2 + 2x + 1)(x^2 + 2x + 1) \\
&= \frac{x^4 + 4x^3 + 6x^2 + 4x + 1}{x^4 + 4x^3 + 6x^2 + 4x + 1}
\end{align*}\]
\[(u+v)(u-v) = u^2 - v^2\]

\[[(x+y)+1][(x+y)-1]\]

\[= (x+y)^2 - 1^2 = (x^2 + 2xy + y^2) - 1\]

\[= (x+v)^2 = u^2 + 2uv + v^2\]

\[(x+y)(x-y)(x^2 + y^2)\]

\[= (x^2 - y^2)(x^2 + y^2) = (x^2)^2 - (y^2)^2\]

\[= x^4 - y^4\]
Factoring

Definitions:
The process of writing a polynomial as a product is called **factoring**. If a polynomial cannot be factored using integer coefficients, it is called **prime** or **irreducible** over the integers. A polynomial is **completely factored** when each of its factors is **prime**.
Examples: Factor completely.

\[ 4x^3 - 6x^2 + 12x \quad \text{(common factors)} \]

\[ = 2x (2x^2 - 3x + 6) \]

\[ 49 - 9y^2 \quad \text{(difference of squares)} \]

\[ = 7^2 - (3y)^2 = (7-3y)(7+3y) \]

\[ \text{or } (7+3y)(7-3y) \]

\[ 25 - (z + 5)^2 \]

\[ = 5^2 - (z+5)^2 = [5-(z+5)][5+(z+5)] \]

\[ = [5-z-5][5+z+5] \]

\[ = -z(z+10) \]
\[ 9x^2 - 12x + 4 = (3x - 2)^2 \]  

\[ z^3 + 125 = (z + 5)^3 = (z + 5)(z^2 - 5z + 25) \]  

\[ t^2 - t - 6 = (t - 3)(t + 2) \]  

\[ x^3 - 27 = (x - 3)(x^2 + 3x + 9) \]
\[2x^2 - x - 21 = (2x - 7)(x + 3)\]

check: \[2x^2 + 6x - 7x - 21\]

\[24 + 5z - z^2 = (8 - z)(3 + z)\]

check: \[24 + 8z - 3z - z^2\]
More examples: Factoring by grouping and combinations.

\[ x^3 + 5x^2 - 5x - 25 \]
\[ = x^2(x+5) - 5(x+5) \]
\[ = (x+5)(x^2 - 5) \]
\[ \text{Check:} \]
\[ (x+5)(x^2 - 5) = x^3 - 5x + 5x^2 - 25 \]

\[ 5x^3 + 40 \]
\[ = 5(x^3 + 8) = 5(x^3 + 2^3) \]
\[ = 5(x+2)(x^2 - 2x + 4) \]

(combinations)
\[5x^3 - 10x^2 + 3x - 6\]
\[= 5x^2(x-2) + 3(x-2)\]
\[= (x-2)(5x^2 + 3) \quad \text{or} \quad (5x^2 + 3)(x-2)\]

\[12x^2 - 48\]
\[= 12(x^2 - 4)\]
\[= 12(x-2)(x+2)\]

\[4x^2 - 36\]
\[= 4(x^2 - 9)\]
\[= 4(x-3)(x+3)\]
P.4 Rational Expressions

Definitions:
The set of real numbers for which an expression is defined is its domain. Two expressions are equivalent if they yield the same value for all numbers in their domain.

The quotient of two algebraic expressions is a fractional expression. The quotient of two polynomials is a rational expression.

\[
\frac{\sqrt{x^2 + 4}}{3x^2 - 5}
\]

\[
\frac{X^2 - 4}{X^2 - 5X + 6}
\]

fractional expression

fractional but also rational expression

(ratio of polynomials)
Examples:

Find the domain:

\[ 2x^2 - 5x - 2 \quad \text{domain: all real numbers or } (-\infty, \infty) \]

\[ 6x^2 - 9, x > 0 \quad \text{domain: } (0, \infty) \quad \text{restriction} \]

exclude from domain:
1. any number that results in 0 in denominator
2. any number that results in a neg. no. under \( \sqrt{1} \)

\[ \frac{x+1}{2x+1} \]  \( x \neq -\frac{1}{2} \) or \((-\infty, -\frac{1}{2}) \) and \((-\frac{1}{2}, \infty)\)

\[ \sqrt{6-x} \]  \( 6-x > 0 \iff -x > -6 \iff x < 6 \quad \text{or } (-\infty, 6) \]