12.8 The Counting Principle and Permutations

Counting Principle If a first experiment can be performed in \( M \) distinct ways and a second experiment can be performed in \( N \) distinct ways, then the two experiments in that specific order can be performed in \( MN \) distinct ways.

A permutation is any ordered arrangement of a given set of objects. (Assume repetition is not permitted.)
Example:

A bag contains 4 marbles:
1 Red; 1 Blue; 1 Yellow; 1 Green

How many arrangements (permutations) are there of these marbles (order counts and no repetitions are allowed).

1st 2nd 3rd 4th
R Y G B
B Y G R
Y B G R
G R B Y

etc.
RBYG  BRYG  YRBG  CRBY
RBGY  BRGY  YRGB  GRYB
RYBG  BYRG  YBRC  GBRY
RYGB  BYGR  YBGR  GBYR
RBGY  BGYR  YGRB  GYRB
RYGB  GBYR  YGBR  GYBR
\[ n = 4 \]
There should be 4! permutations
\[ 4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24 \]
The **number of permutations** of $n$ distinct items is $n$ factorial, $n!$, where

$$n! = n(n-1)(n-2)\ldots(3)(2)(1)$$

$$0! = 1$$

$_n^P_r = \text{the number of permutations of } n \text{ objects taken } r \text{ at a time.}$

$$\frac{n!}{(n-r)!}$$

In the previous example, suppose you chose only 2 of the 4 marbles. How many permutations are there?

\[ R \ B \quad B \ R \quad Y \ R \quad G \ R \]

\[ R \ Y \quad B \ Y \quad Y \ B \quad G \ B \]

\[ R \ G \quad B \ G \quad Y \ G \quad G \ Y \]

\[ \frac{4!}{(4-2)!} = \frac{4!}{2!} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1} = 12 \]
Examples:

\[ _5P_3 = \frac{5!}{(5-3)!} = \frac{5!}{2!} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1} = 60 \]

\[ _{12}P_{10} = \frac{12!}{(12-10)!} = \frac{12!}{2!} = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1} = 239,580,800 \]

\[ _{12}P_2 = \frac{12!}{(12-2)!} = \frac{12!}{10!} = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 12 \cdot 11 = 132 \]
Examples:

1. **Daily Double** The daily double at most race tracks consists of selecting the winning horse in both the first and second races. If the first race has seven entries and the second race has eight entries, how many daily double tickets must you purchase to guarantee a win? $7 \times 8 = 56$
2. **Social Security Numbers** A social security number consists of nine digits. How many different social security numbers are possible if repetition of digits is permitted?

\[10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 = 10^9 = 1,000,000,000\]

1 billion
3. **Arranging Pictures** The six pictures shown (named) are to be placed side by side along a wall.

BIRD  DOG  CAT  GIRAFFE  TURTLE  LION

In how many ways can they be arranged from left to right if:

a) they can be arranged in any order?

\[
\frac{6!}{6} \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 6! = 720
\]

b) the bird must be on the far left?

\[
\frac{5!}{6} \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120
\]
3. **Arranging Pictures** The six pictures shown (named) are to be placed side by side along a wall.

   **BIRD** **DOG** **CAT** **GIRAFFE** **TURTLE** **LION**

   In how many ways can they be arranged from left to right if

   c) the bird must be on the far left and the giraffe must be next to the bird?

   \[
   \frac{1 \cdot 1 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{B \ G} = 24
   \]

   d) a four-legged animal must be on the far right?

   \[
   5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 5 = 600
   \]
4. A teacher decides to give 6 different prizes to 6 of the 30 students in her class. In how many ways can she do so?

\[
30P_6 = \frac{30!}{(30-6)!} = \frac{30!}{24!} = \ldots
\]

(use calculator)

= 427,518,000

Homework (Section 12.8)
### 100 fastest growing counties

#### Population Estimates for the 100 Fastest Growing U.S. Counties with 10,000 or more Population in 2004:
*April 1, 2000 to July 1, 2004*

<table>
<thead>
<tr>
<th>Geographic Area</th>
<th>Population estimates</th>
<th>Change, 2000 to 2004</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>July 1, 2004</td>
<td>April 1, 2000</td>
</tr>
<tr>
<td>Loudoun County, VA</td>
<td>239,156</td>
<td>169,599</td>
</tr>
<tr>
<td>Flagler County, FL</td>
<td>69,005</td>
<td>49,632</td>
</tr>
<tr>
<td>Douglas County, CO</td>
<td>237,963</td>
<td>175,766</td>
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<tr>
<td>Rockwall County, TX</td>
<td>58,250</td>
<td>43,083</td>
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<tr>
<td>Forsyth County, GA</td>
<td>101,865</td>
<td>90,407</td>
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<tr>
<td>Henry County, GA</td>
<td>159,506</td>
<td>119,404</td>
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<tr>
<td>Kendall County, IL</td>
<td>72,548</td>
<td>54,544</td>
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<tr>
<td>Newton County, GA</td>
<td>91,524</td>
<td>62,001</td>
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<tr>
<td>Lincoln County, SD</td>
<td>31,437</td>
<td>24,147</td>
</tr>
<tr>
<td>Paulding County, GA</td>
<td>105,536</td>
<td>81,647</td>
</tr>
<tr>
<td>Delaware County, OH</td>
<td>142,503</td>
<td>109,989</td>
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<tr>
<td>Scott County, MN</td>
<td>114,794</td>
<td>89,498</td>
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<td>Collin County, TX</td>
<td>627,938</td>
<td>491,774</td>
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<tr>
<td>Osceola County, FL</td>
<td>219,544</td>
<td>172,492</td>
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<tr>
<td>Williamson County, TX</td>
<td>317,938</td>
<td>249,967</td>
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</tbody>
</table>
Following Benford's Law, or Looking Out for No. 1

By Malcolm W. Browne

(From The New York Times, Tuesday, August 4, 1998)

Dr. Theodore P. Hill asks his mathematics students at the Georgia Institute of Technology to go home and either flip a coin 200 times and record the results, or merely pretend to flip a coin and fake 200 results. The following day he runs his eye over the homework data, and to the students' amazement, he easily fingers nearly all those who faked their tosses.

"The truth is," he said in an interview, "most people don't know the real odds of such an exercise, so they can't fake data convincingly."

There is more to this than a classroom trick.

Dr. Hill is one of a growing number of statisticians, accountants and mathematicians who are convinced that an astonishing mathematical theorem known as Benford's Law is a powerful and relatively simple tool for pointing suspicion at frauds, embezzlers, tax evaders, sloppy accountants and even computer bugs.
The income tax agencies of several nations and several states, including California, are using detection software based on Benford's Law, as are a score of large companies and accounting businesses.

Benford's Law is named for the late Dr. Frank Benford, a physicist at the General Electric Company. In 1938 he noticed that pages of logarithms corresponding to numbers starting with the numeral 1 were much dirtier and more worn than other pages.

(A logarithm is an exponent. Any number can be expressed as the fractional exponent -- the logarithm -- of some base number, such as 10. Published tables permit users to look up logarithms corresponding to numbers, or numbers corresponding to logarithms.)
Logarithm tables (and the slide rules derived from them) are not much used for routine calculating anymore; electronic calculators and computers are simpler and faster. But logarithms remain important in many scientific and technical applications, and they were a key element in Dr. Benford's discovery.

Dr. Benford concluded that it was unlikely that physicists and engineers had some special preference for logarithms starting with 1. He therefore embarked on a mathematical analysis of 20,229 sets of numbers, including such wildly disparate categories as the areas of rivers, baseball statistics, numbers in magazine articles and the street addresses of the first 342 people listed in the book "American Men of Science." All these seemingly unrelated sets of numbers followed the same first-digit probability pattern as the worn pages of logarithm tables suggested. In all cases, the number 1 turned up as the first digit about 30 percent of the time, more often than any other.
Benford's law predicts a decreasing frequency of first digits, from 1 through 9. Every entry in data sets developed by Benford for numbers appearing on the front pages of newspapers, by Mark Nigrini of 3,141 county populations in the 1990 U.S. Census and by Eduardo Ley of the Dow Jones Industrial Average from 1990-93 follows Benford's law within 2 percent.
Benford’s law can be used to test for fraudulent or random-guess data in income tax returns and other financial reports. Here the first significant digits of true tax data taken by Mark Nigrini from the lines of 169,662 IRS model files follow Benford’s law closely. Fraudulent data taken from a 1995 King’s County, New York, District Attorney’s Office study of cash disbursement and payroll in business do not follow Benford’s law. Likewise, data taken from the author’s study of 743 freshmen’s responses to a request to write down a six-digit number at random do not follow the law. Although these are very specific examples, in general, fraudulent or concocted data appear to have far fewer numbers starting with 1 and many more starting with 6 than do true data.

<table>
<thead>
<tr>
<th>Benford’s law</th>
<th>20.1</th>
<th>17.6</th>
<th>12.5</th>
<th>9.7</th>
<th>7.9</th>
<th>6.7</th>
<th>5.8</th>
<th>5.1</th>
<th>4.6</th>
</tr>
</thead>
<tbody>
<tr>
<td>true tax data</td>
<td>30.5</td>
<td>17.6</td>
<td>12.5</td>
<td>9.7</td>
<td>7.9</td>
<td>6.7</td>
<td>5.8</td>
<td>5.1</td>
<td>4.5</td>
</tr>
<tr>
<td>fraudulent data</td>
<td>0</td>
<td>1.9</td>
<td>0</td>
<td>9.7</td>
<td>81.2</td>
<td>23.3</td>
<td>1.0</td>
<td>2.9</td>
<td>6</td>
</tr>
<tr>
<td>random-guess data</td>
<td>14.7</td>
<td>10.0</td>
<td>19.4</td>
<td>13.3</td>
<td>9.7</td>
<td>15.7</td>
<td>12.9</td>
<td>8.4</td>
<td>5.8</td>
</tr>
</tbody>
</table>
Some of the tests based on Benford’s Law are so complex that they require a computer to carry out. Others are surprisingly simple, just finding too few ones and too many sixes in a sequence of data to be consistent with Benford's Law is sometimes enough to arouse suspicion of fraud.

Robert Burton, the chief financial investigator for the Brooklyn District Attorney, recalled in an interview that he had read an article by Dr. Nigrini that fascinated him.

"He had done his Ph.D. dissertation on the potential use of Benford's Law to detect tax evasion, and I got in touch with him in what turned out to be a mutually beneficial relationship," Mr. Burton said. "Our office had handled seven cases of admitted fraud, and we used them as a test of Dr. Nigrini's computer program. It correctly spotted all seven cases as "involving probable fraud."

One of the earliest experiments Dr. Nigrini conducted with his Benford's Law program was an analysis of President Clinton's tax return. Dr. Nigrini found that it probably contained some rounded-off estimates rather than precise numbers, but he concluded that his test did not reveal any fraud.

The fit of number sets with Benford's Law is not infallible.

"You can't use it to improve your chances in a lottery," Dr. Nigrini said. "In a lottery someone simply pulls a series of balls out of a jar, or something like that. The balls are not really numbers; they are labeled with numbers, but they could just as easily be labeled with the names of animals. The numbers they represent are uniformly distributed, every number has an equal chance, and Benford's Law does not apply to uniform distributions."
Another problem Dr. Nigrini acknowledges is that some of his tests may turn up too many false positives. Various anomalies having nothing to do with fraud can appear for innocent reasons.

For example, the double digit 24 often turns up in analyses of corporate accounting, biasing the data, causing it to diverge from Benford's Law patterns and sometimes arousing suspicion wrongly, Dr. Nigrini said. "But the cause is not real fraud, just a little shaving. People who travel on business often have to submit receipts for any meal costing $25 or more, so they put in lots of claims for $24.90, just under the limit. That's why we see so many 24's."