Section 5.1

#13. \( (1, x) \)

\[ (1, y) + (1, y') = (1, y + y') \]

\[ k(1, y) = (1, ky) \]

Axioms \( \text{(1) + (6)} \)

\[ \Theta \overrightarrow{v} + \overrightarrow{u} = \overrightarrow{u} + \overrightarrow{v} \]

\[ \overrightarrow{u} + \overrightarrow{v} = (1, y + y') = (1, y') + (1, y) = \overrightarrow{u} + \overrightarrow{v} \]

\[ \overrightarrow{u} + (\overrightarrow{v} + \overrightarrow{w}) = (\overrightarrow{u} + \overrightarrow{v}) + \overrightarrow{w} \]

\[ \overrightarrow{u} + (1, y) + ((1, y') + (1, y'')) = (1, y) + (1, y + y'') \]

\[ = (1, y + (y + y'')) = (1, (y + y') + y'') - (1, \overrightarrow{v} + \overrightarrow{w}) + \overrightarrow{w} \]
④ \[ \overrightarrow{\mathbf{u}} + \overrightarrow{\mathbf{v}} = \overrightarrow{\mathbf{u} + v} \]

Let \( \overrightarrow{\mathbf{v}} = (1, 0) \)

\[
(1, 0) + (1, y) = (1, y) = \overrightarrow{\mathbf{u}}
\]

\[
(1, y) + (1, 0) = (1, y) = \overrightarrow{\mathbf{v}}
\]

⑤ \(-\overrightarrow{\mathbf{u}} = (1, -y)\)

\[
\overrightarrow{\mathbf{u}} + (-\overrightarrow{\mathbf{u}}) = (1, y) + (1, -y) =
\]

\[
(1, 0) = \overrightarrow{0}
\]  

⑦ \[ k(\overrightarrow{\mathbf{u}} + \overrightarrow{\mathbf{v}}) = k\overrightarrow{\mathbf{u}} + k\overrightarrow{\mathbf{v}} \]

\[
k\overrightarrow{\mathbf{u}} + k\overrightarrow{\mathbf{v}} = (1, ky) + (1, ky') = (1, ky + ky')
\]

\[
= (1, k(y + y')) = k(\overrightarrow{\mathbf{u}} + \overrightarrow{\mathbf{v}})
\]
\( (k + \lambda) \vec{u} = k \vec{u} + \lambda \vec{u} \)

\[
(k + \lambda) \vec{u} = (1, (k + \lambda) y) = (1, ky + \lambda y) \\
= (1, ky) + (1, \lambda y) = k \vec{u} + k \vec{v}
\]

\( k (\lambda \vec{u}) = (k \lambda) \vec{u} \)

\[
k (1, ky) = (1, k(\lambda y)) = (1, (k \lambda) y) \\
= (k \lambda) \vec{u} \checkmark
\]

\( 1 \vec{u} = 1 (1, y) = (1, y) = \vec{u} \checkmark \)

It is a vector space.
5.2 **Subspaces**

**Definition.** A subset $W$ of a vector space $V$ is called a **subspace** of $V$ if $W$ is itself a vector space under the addition and multiplication defined on $V$.

**Example.** Lines and planes through the origin are subspaces of $\mathbb{R}^3$.

$W$ "inherits" axioms 2, 3, 7, 8, 9, 10 from $V$. Theorem 5.2.1 shows that axioms 4 and 5 hold if axioms 1 to 6 hold.
Theorem 5.2.1: If \( W \) is a set of one or more vectors from a vector space \( V \), then \( W \) is a subspace of \( V \) iff. the following conditions hold:

a.) If \( \vec{u}, \vec{v} \in W \), then \( \vec{u} + \vec{v} \in W \).
(closure under addition)

b.) If \( \vec{u} \in W \), \( k \) is any scalar, \( k\vec{u} \in W \).
(closure under scalar multiplication)

Proof: Let \( k = 0 \) \( 0\vec{u} = \vec{0} \in W \) (Ax.4)
Let \( k = -1 \) \( -1\vec{u} = -\vec{u} \in W \) (Ax.5)
Note:
Every vector space \( V \) has at least 2 subspaces: \( V \) and \( \{0\} \).

\( \mathbb{R}^3 \): subspaces are \( \mathbb{R}^3, \{0\} \), lines and planes through the origin.

\( \mathbb{R}^2 \): subspaces are \( \mathbb{R}^2, \{0\} \), lines through the origin.
$F(\mathbb{R}, \mathbb{R})$ : set of all real-valued functions defined on $(\mathbb{R}, \mathbb{R})$

Some subspaces:

$C(\mathbb{R}, \mathbb{R})$ set of all continuous real-valued functions defined on $(\mathbb{R}, \mathbb{R})$

$C'(\mathbb{R}, \mathbb{R})$ set of all functions with continuous first derivatives on $(\mathbb{R}, \mathbb{R})$

$C'$ is a subspace of $C$ and also a subspace of $F$
$C^m$ is the subspace with cont. $m$th derivatives.

$C^\infty$ is the subspace with cont. derivatives of all orders.

$P_n$ is the set of polynomials of degree $\leq n$ with real coeff.

$P_3$ is a subspace of $F(-\infty,\infty)$, a subspace of $C(-\infty,\infty)$, a subspace of $C'(-\infty,\infty)$, a subspace of $C^\infty(-\infty,\infty)$.

Repeat all of this starting with $F[a, b]$ or $F(a, b)$. 
For a set of $m$ linear equations in $n$ unknowns $(Ax = b)$, a vector $\vec{s} = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}$ in $\mathbb{R}^n$ is a solution vector of the system if $x_1 = s_1, \ldots, x_n = s_n$.

Theorem 5.2.2 For $Ax = \vec{0}$ (homogeneous system) the set of solution vectors is a subspace of $\mathbb{R}^n$ (means in $n$ unknowns).
Let $A\mathbf{x} = \mathbf{0}$ be an $m \times n$ homogeneous system; let $\mathbf{s}$ and $\mathbf{s}'$ be any solns; let any scalar $k$.

$\Rightarrow A\mathbf{s} = \mathbf{0}$ and $A\mathbf{s}' = \mathbf{0}$

1. $A(\mathbf{s} + \mathbf{s}') = A\mathbf{s} + A\mathbf{s}' = \mathbf{0} + \mathbf{0} = \mathbf{0}$

   $\Rightarrow \mathbf{s} + \mathbf{s}'$ is also a soln. vector.

   (Closure under addition)

2. $A(k\mathbf{s}) = k(A\mathbf{s}) = k\mathbf{0} = \mathbf{0}$

   $\Rightarrow k\mathbf{s}$ is also a soln. vector.

   (Closure under scalar multiplication)

**Definition:** If $W$ is the subspace of all solution vectors of $A\mathbf{x} = \mathbf{0}$, $W$ is called the solution space of the system.
Definition. A vector $\mathbf{w}$ is called a linear combination of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_r$ if $\mathbf{w}$ can be expressed in the form $\mathbf{w} = k_1 \mathbf{v}_1 + \ldots + k_r \mathbf{v}_r$ where $k_i$'s are scalars.

Ex. (from pp. 219-221) Determine whether the given set is a subspace of the given vector space.

1b. $W =$ set of all vectors of the form $(a, 1, 1)$, subspace of $\mathbb{R}^3$. 

$(a, 1, 1) + (b, 1, 1) = (a + b, 2, 2)$

$k + 1 \cdot (a, 1, 1) = (ka, k, k)$ not in $W$

$k \cdot (a, 1, 1) = (ka, k, k)$ not in $W$

not a subspace
3b. \( W = \text{set of all polynomials of the form} \ a_0 + a_1 x + a_2 x^2 + a_3 x^3 \ \text{for which} \ a_0 + a_1 + a_2 + a_3 = 0 \) (subspace of \( P_3 \))

Let \( p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \in W \)
\( q(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3 \in W \)

\[ (p + q)(x) = p(x) + q(x) \]
\[ = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + b_0 + b_1 x + b_2 x^2 + b_3 x^3 \]
\[ = (a_0 + b_0) + (a_1 + b_1) x + (a_2 + b_2) x^2 + (a_3 + b_3) x^3 \]
\[ \stackrel{(a_0 + b_0) + (a_1 + b_1) + (a_2 + b_2) + (a_3 + b_3)}{=} (a_0 + b_0) + (a_1 + b_1) + (a_2 + b_2) + (a_3 + b_3) = 0 + 0 = 0 \]
\[ \Rightarrow (p + q)(x) \in W. \]
\( \Theta (k p) (x) = k p (x) = k (a_0 + \ldots + a_3 x^3) = \frac{ka_0 + ka_1 + ka_2 x + ka_3 x^3}{ka_0 + ka_1 + ka_2 + ka_3} = k(a_0 + a_1 + a_2 + a_3) = k \cdot 0 = 0 \)

\[ \Rightarrow (kp)(x) \in W \]

\[ \Rightarrow W \text{ is a subspace of } P_3. \]

46. \( W = \text{ set of all real-valued functions for which } f(0) = 0 \)
(Subspace of \( F(-\infty, \infty) \))

\[ f, g \in W; k \text{ is any scalar} \]

\( f + g \in W \)

\( (f + g)(0) = f(0) + g(0) = 0 + 0 = 0 \)

\( kf \in W \)

\[ (kf)(0) = k f(0) = k \cdot 0 = 0 \]

\( W \text{ is a subspace of } F(-\infty, \infty) \)
Theorem 5.2.3. If \( \vec{v}_1, ..., \vec{v}_r \) are vectors in a vector space \( V \), then:

1. The set \( W \) of all linear combinations of \( \vec{v}_1, ..., \vec{v}_r \) is a subspace of \( V \).

2. \( W \) is the smallest subspace of \( V \) that contains \( \vec{v}_1, ..., \vec{v}_r \) in the sense that every subspace of \( V \) that contains \( \vec{v}_1, ..., \vec{v}_r \) contains \( W \).

Definition. If \( S = \{ \vec{v}_1, ..., \vec{v}_r \} \) is a set of vectors in a vector space \( V \), then the subspace \( W \) of \( V \) containing all linear combinations of the vector set \( S \) is called the space spanned by \( \vec{v}_1, ..., \vec{v}_r \); we say \( \vec{v}_1, ..., \vec{v}_r \) span \( W \), write \( W = \text{span}(S) = \text{span}\{\vec{v}_1, ..., \vec{v}_r\} \).
Theorem 5.2.4. If \( S = \{ \vec{v}_1, ..., \vec{v}_{r-3} \} \) and 
\( S' = \{ \vec{w}_1, ..., \vec{w}_{r-3} \} \) are two sets of vectors
in a vector space \( V \), then
\[
\text{span} \{ \vec{v}_1, ..., \vec{v}_{r-3} \} = \text{span} \{ \vec{w}_1, ..., \vec{w}_{r-3} \}
\]
iff. each vector in \( S \) is a linear combination of vectors in \( S' \) and
each vector in \( S' \) is a linear combination of the vectors in \( S \).
ex. $14d$. Is $(-4, 6, -13, 4)$ in the span of 
$v_1 = (2, 1, 0, 3)$, $v_2 = (3, -1, 5, 2)$, and 
$v_3 = (-1, 0, 2, 1)$?

Can we find $a, b, c$ such that 
$(-4, 6, -13, 4) = a \vec{v}_1 + b \vec{v}_2 + c \vec{v}_3$

$= a (2, 1, 0, 3) + b (3, -1, 5, 2)$

$= c (-1, 0, 2, 1)$

$2a + 3b - c = -4$
$a - b = 6$
$5b + 2c = -13$
$3a + 2b + c = 4$

\[ \begin{bmatrix} 2 & 3 & -1 & -4 \\ 1 & -1 & 0 & 6 \\ 0 & 5 & 2 & -13 \\ 3 & 2 & 1 & 4 \end{bmatrix} \text{Ref} \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \]

$a = 3$
$b = -3$
$c = 1$
Is $(2, 0, -2, 3)$ in span $\{v_1, v_2, v_3\}$?

Is $(2, 0, -2, 3) = a v_1 + b v_2 + c v_3$ for some $a, b, c$.

augmented matrix

\[
\begin{bmatrix}
2 & 3 & -1 & 2 \\
-1 & 0 & 0 & 0 \\
0 & 5 & 2 & -2 \\
3 & 2 & 1 & 3
\end{bmatrix}
\xrightarrow{\text{ref}}
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

0 = 1 (X)

$\Rightarrow$ no solution