Orthogonality of functions: One of the big discoveries in modern mathematics was that it is very fruitful to extend some of the geometric notions of vectors to functions - for example, taking the dot product of two functions or asking whether two functions are orthogonal, or attempting to project a function onto a second function (or onto a “plane” of functions). You are introduced to these important concepts in the problems below.

1. We say that two functions are orthogonal on \([a, b]\) if 
\[
\int_a^b f(x)g(x)\,dx = 0.
\]
   a. Show that the two functions \(x^2\) and \(-3x^5 + x\) are orthogonal on \([-b, b]\) for any positive \(b\).
   b. For positive integers \(k\) and \(n \neq k\), show that \(\sin(nx)\) and \(\sin(kx)\) are orthogonal on the interval \([-\pi, \pi]\). You may need the product to sum identity
   \[
   2\sin(\alpha)\sin(\beta) = \cos(\alpha - \beta) - \cos(\alpha + \beta).
   \]

2. Jean Baptiste Joseph Fourier began his studies by training for the priesthood. In spite of this first calling, his interest in mathematics remained intense, and in 1789 he wrote, “Yesterday was my 21st birthday, and at the age of Newton and Pascal had already acquired many claims to immortality.” He did not take his religious vows but continued his mathematical research while teaching. Fourier was renowned as an outstanding teacher, but it was not until 1804-1807 that he did his important mathematical work on the theory of heat. During that time, he made the remarkable observation that odd functions \([\text{that is, } f(-x) = -f(x)]\) can be approximated by
   \[
   f(x) \approx b_1 \sin(x) + b_2 \sin(2x) + \ldots + b_n \sin(nx)
   \]
   \[
   \approx \sum_{k=1}^{n} b_k \sin(kx) = S_n(x)
   \]
   This formula reminds us of Taylor approximations; in fact some call \(S_n(x)\) a trigonometric polynomial. However this expansion is physically more meaningful because it breaks \(f(x)\) down into frequency components, where these components can represent things like pitch in sound.
a. Prove that the formula for the coefficients of \( S_n(x) \), \( b_k \), is

\[
b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx) \, dx = \frac{2}{\pi} \int_{0}^{\pi} f(x) \sin(kx) \, dx
\]

Assume that \( S_n(x) = f(x) \left( \sum_{k=1}^{n} b_k \sin(kx) = f(x) \right) \) and multiply both sides by \( \sin(jx) \) and then take the integral of both sides over the interval \([ -\pi, \pi ]\). Use the result from 1b. You will get a formula in terms of \( b_j \).

Even if you can’t do part a, you can still complete parts b, c, d, and e.

b. Show that the sine (Fourier) expansion of the simple function \( f(x) = x \) is

\[
x = 2\sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} \sin(kx)
\]

c. Carefully graph \( y = x \) and \( S_n(x) \) for \( n = 1, 3, 5, \) and 7.

d. For the function \( f(x) = xe^{x^2} \), find \( S_n(x) \) for \( n = 1, 3, 5, \) and 7.

e. Carefully graph \( S_n(x) \) for \( n = 1, 3, 5, \) and 7 and the function \( f(x) = xe^{x^2} \).