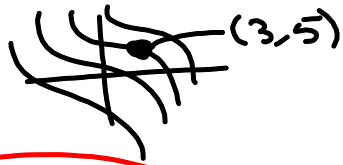


One-Parameter Family of Solutions and Particular Solutions

Example 9:

(a) Find the one-parameter family of solutions to the DE

$$\frac{dx}{dt} = 5 - t^2$$

$$\int dx = \int 5 - t^2 \, dt \Rightarrow x(t) = 5t - \frac{t^3}{3} + C$$

(b) Find the particular solution for the initial condition x(3) = 5.

$$5 = 5 \cdot 3 - \frac{3^3}{3} + C$$

$$5 = 6 + C$$

$$-1 = C$$

$$x(t) = 5t - \frac{t^3}{3} - 1$$

explicit sol'n

(19)

DE: $\frac{dx}{dt} \stackrel{?}{=} (x-1)(1-2x)$ dep, indep

Verify that $\ln\left(\frac{2x-1}{x-1}\right)=t$ is a sol'n

Sol'n: implicit sol'n: $\ln\left(\frac{2x-1}{x-1}\right)=t$

(explicit $x(t)=$)

diff implicitly

$$\ln(2x-1)-\ln(x-1)=t$$

$$\frac{1}{2x-1}\cdot 2\cdot\frac{dx}{dt}-\frac{1}{x-1}\cdot\frac{dx}{dt}=1$$

$$\frac{dx}{dt}\left[\frac{2}{2x-1}\left(\frac{x-1}{x-1}\right)-\frac{1}{x-1}\frac{(2x-1)}{(2x-1)}\right]=1$$

$$\frac{(2x-1)(x-1)}{-1}\frac{dx}{dt}\left[\frac{-1}{\cancel{(2x-1)(x-1)}}\right]=1\,\frac{(2x-1)(x-1)}{-1}$$

So $\frac{dx}{dt}=-(2x-1)(x-1)$

Sub. into DE & check:

DE: LHS $=\frac{dx}{dt}=-(2x-1)(x-1)$

$=(1-2x)(x-1)$

$=$ RHS ✓

So $\ln\left(\frac{2x-1}{x-1}\right)=t$ is a sol'n

1.2 Initial-Value Problems

Example 1: Use the fact that $y = c_1 \cos t + c_2 \sin t$ is a two-parameter family of solutions of the DE $y'' + y = 0$ to find a solution with initial conditions: $y\left(\frac{\pi}{2}\right) = 1$. $y'(\pi/2) = 1$

Verify y is a sol'n to DE.

diff: $y' = -c_1 \cdot \sin t + c_2 \cos t$
$y'' = -c_1 \cdot \cos t - c_2 \sin t$ } sub into DE

DE: $\text{LHS} = y'' + y = \left(-\cancel{c_1 \cos t} - \cancel{c_2 \sin t}\right) + \left(\cancel{c_1 \cos t} + \cancel{c_2 \sin t}\right)$
$= 0 = \text{RHS}$ ✓

So y is a sol'n to the DE.

$1 = y'(\pi/2) = -c_1 \cdot 1 + c_2 \cdot 0 \Rightarrow c_1 = -1$

$1 = y(\pi/2) = -\overset{0}{\cos(\pi/2)} + c_2 \overset{1}{\sin(\pi/2)}$

$c_2 = 1$

$$\boxed{y = -1 \cdot \cos t + 1 \cdot \sin t}$$

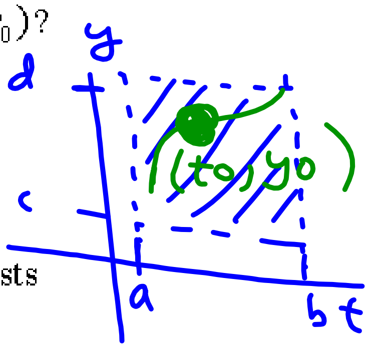
Existence and Uniqueness:

Existence: Does the DE $\frac{dy}{dx} = f(t, y)$ possess solutions? Do any of the solutions pass through the point (t_0, y_0)

x = t

initial cond.

Uniqueness: When can we be certain that there is precisely one solution curve passing through the point (t_0, y_0)?



Existence of a Unique Solution Theorem:
Let R be a rectangular region in the xy-plane defined by $a \leq x \leq b; c \leq y \leq d$ that contains the point (t_0, y_0) in its interior. If $f(x, y)$ and $\frac{\delta f}{\delta y}$ are continuous on R, then there exists some interval $I_0 : t_0 - h < t < t_0 + h,$ for $h > 0$ contained in $a \leq t \leq b$ and a unique function $y(t)$ defined on I_0 that is a solution of the initial value problem.

1 2

Example 2: Determine a region of the ~~xy~~ t y-plane for which the given DE will have a unique solution whose graph passes through a point (t_0, y_0) in the region.

(a) $t\dfrac{dy}{dt} = y$

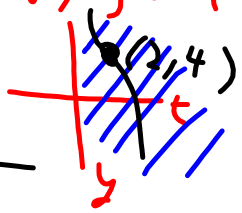
$$\frac{t\frac{dy}{dt}}{t} = \frac{y}{t} \Rightarrow \frac{dy}{dt} = \frac{y}{t} = f(t,y)$$

① $f(t,y) = \dfrac{y}{t}$ is cont's for all (t,y) except where $t=0$

② $\dfrac{\partial f}{\partial y} = \dfrac{1}{t}$ is also cont's for all (t,y), where $t \neq 0$

if $R = \{(t,y) \mid t>0\}$ & if (t_0, y_0) in R

DE has a unique sol'n on R



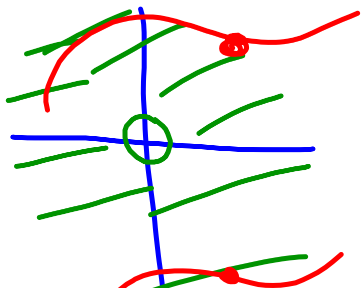
(b) $(t^2+y^2)y' = y^2$

$$\frac{dy}{dt} = \frac{y^2}{t^2+y^2} = f(t,y)$$ is cont's for all (t,y) except where $(t,y)=(0,0)$

$$\frac{\partial f}{\partial y} = \frac{(t^2+y^2)\cdot 2y - y^2(2y)}{(t^2+y^2)^2} = \frac{2yt^2}{(t^2+y^2)^2}$$

is also cont's for all (t,y) except $(t,y) \neq (0,0)$

$R = \{(t,y) \mid (t,y) \neq (0,0)\}$



if i.c. is $(2,4)$

there is a unique sol'n to DE through $(2,4)$

if ic is $(0,0)$, i can't guarantee a unique sol'n.

(not in R)

Example 3: Determine whether the Existence Theorem guarantees that the DE $y' = \sqrt{y^2 - 9}$ possesses a unique solution that goes through the point

(a) (-1, 1)

(b) (5,3)

T = temp of coffee

t = time, minutes

1.3 Differential Equations as Models

Example 1 (Newton's Law Of Cooling):

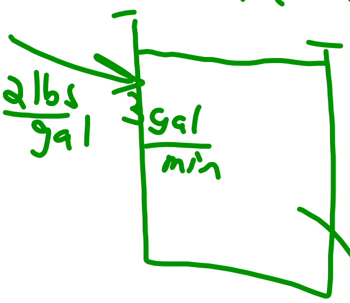
The rate at which coffee cools is directly proportional to the difference between the temperature of the coffee T and the temperature of the surrounding medium T_m.

$$\boxed{\frac{dT}{dt} = k \cdot (T - T_m)}$$

Example 2A (Mixtures):

Suppose that a large mixing tank initially holds 300 gallons of water in which 50 pounds of salt have been dissolved. Another brine solution (with 2 lbs of salt per gallon) is being pumped in at a rate of 3 gal/min, the solution is well stirred and pumped out at the same rate. Determine a DE for the amount of salt A(t) at time t.

$A(0) = 50$ lbs $A(t) =$ lbs of salt at time t



$R_{in} = 6$ lbs/min

$R_{out} = \frac{A \text{ lbs}}{300 \text{ gal}} \cdot \frac{3 \text{ gal}}{\text{min}} = \frac{A}{100}$ lb/min

$\frac{A \text{ lbs}}{300 \text{ gal}} \cdot \frac{3 \text{ gal}}{\text{min}}$

$$\frac{dA}{dt} = R_{in} - R_{out}$$

$$\boxed{\frac{dA}{dt} = 6 - \frac{A}{100}}$$ with $A(0) = 50$

Example 2B: What if the solution in 2A is pumped out at a slower rate of 2 gallons per minute ?

$$\frac{dA}{dt} =$$

Example 2C: What if the solution in 2A is pumped out at a faster rate of 3.5 gallons per minute ?

WED: Ch1 (1.2, 1.3) book HW

Quiz 1 Due @ 11:15 Wed.