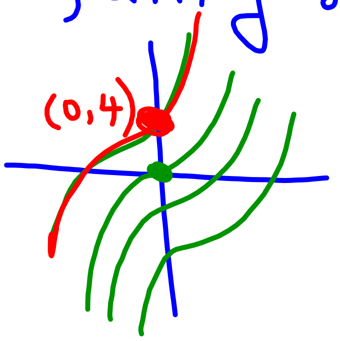
$$\frac{dy}{dx} = x^2$$

$$\int dy = \int x^2 \, dx$$

$$y = \frac{x^3}{3} + C$$

family of sol'ns



initial condition

$$y(0) = 4$$

$$4 = \frac{0^3}{3} + C$$

$$4 = C$$

sol'n: $y = \frac{x^3}{3} + 4$

Chapter 1

An Introduction to Differential Equations

1.1 Definitions and Terminology

$\frac{d^2h}{dt^2}$ h=position

Example 1A (Free Fall)

Newton's Second Law of Motion states that an object's mass m times its acceleration is equal to the total force acting on the object. If we let h represent the height of the object, then we can restate Newton's Second Law of Motion as: $\frac{m}{m}\cdot\frac{d^2h}{dt^2} = -\frac{gm}{m}$. If a baseball was dropped from the top of a 100 foot building, then this equation can be simplified to $\boxed{\frac{d^2h}{dt^2} = -32 \text{ ft/s}^2}$. This is an example of a **second order ordinary differential equation.**

Definitions:
An equation containing derivatives of one or more unknown functions with respect to one or more independent variables, is said to be a **differential equation** (DE).

not ord: partial deriv $\frac{\partial f}{\partial x}$ or $\frac{\partial f}{\partial y}$

An **ordinary differential equation** (ODE) contains only ordinary derivatives of one or more dependent variables with respect to a single independent variable.

Example 1: Find the order of the DE

$$\frac{dy}{dx} = x^2 + y^2$$ 1st order DE - Ch 2

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 0$$ 2nd order DE - Ch 4

$$\frac{dy}{dt} + 3\frac{dx}{dt} = t$$ 1st order

A **partial differential equation** (PDE) contains partial derivatives of one or more dependent variables of two or more independent variables.

Example 2

$$\frac{\delta y}{\delta t}+3\frac{\delta y}{\delta s}=t$$ 1st order PDE

$$\frac{\delta^2 u}{\delta x^2}=3\frac{\delta^2 u}{\delta y^2}+2y=0$$ 2nd order PDE

The **order of a differential equation** (ODE or PDE) is the order of the highest derivative in the equation.

Examples 3 Give the order of the following DE's

$$\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + 3 = 0$$ 2nd order ODE

$$\left(\frac{dy}{dx}\right)^2 = 2x + 3$$ 1st order ODE

$$3x\frac{\cancel{dx}}{dx} + 5y^2\frac{dy}{dx} = 0$$

$$3x + 5y^2\frac{dy}{dx} = 0$$ 1st order ODE

A general nth order ordinary differ.rential equation is often written

dep. var

as $F(t, y, y', \ldots, y^{(n)}) = 0$ where $y^{(n)} = \dfrac{d^n y}{dt^n}$.

indep t or x

An ordinary differential equation is said to be **linear** if it can be written in the form

$$\boxed{a_n(t)}\frac{d^n y}{dt^n} + a_{n-1}(t)\frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_1(t)\frac{dy}{dt} + a_0(t)y = g(t).$$

only funct. of indep. var

Note: The dependent variable (y) and all of its derivatives are of the first degree and each coefficient depends only on the independent variable, which is usually either t or x.

Example 4 Determine which of the following DE's are linear and which are nonlinear.

(a) $e^{2t}\frac{dy}{dt} = \sinh t \frac{d^2y}{dt^2}$ Linear

(b) $x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + 2xy = \sin x$ Linear

(c) $\frac{d^3y}{dt^3} = -3t$ Linear **(d)** $y^2\frac{d^2y}{dt^2} = \frac{dy}{dt}$ Non-Linear

Non-Linear

(e) $(\frac{dy}{dx})^2 = 2x$

(f) $y\frac{dy}{dt} = 1, \frac{dy}{dt} = \frac{1}{y}$ Non-Linear

Any function ϕ defined on some interval *I*, which when substituted into a differential equation reduces the equation to an identity, is said to be a **solution** of the equation on the interval. The interval *I* is called the **interval of definition**.

A solution in which the dependent variable is expressed solely in terms of the independent variable and constants is said to be an **explicit solution**. An explicit solution of a differential equation that is identically zero on an interval *I* is called a **trivial solution.**

A relation $G(t, y) = 0$ is said to be an **implicit solution** of an ordinary differential equation on an interval *I* provided there exists at least one function ϕ that satisfies the relation as well as the differential equation on *I*. That is, $G(t, y) = 0$ defines the function implicitly.

Example 5: Under compound interest, the rate of change of a bank account balance *B(t)* is proportional to the size. If $7000 is initially invested, verify that $B(t) = 7000e^{kt}$ is the solutions to the initial value DE $\boxed{\frac{dy}{dt} = kB; \quad B(0) = 7000}$

$$\frac{dB}{dt} = k \cdot B$$

$$B(0) = 7000$$

$$\phi(t) = B(t) = 7000e^{kt}$$

Check: ① $\frac{dB}{dt} = 7000 \cdot e^{kt} \cdot \overbrace{k}^{\text{chain Rule}}$

$$= k\left(7000e^{kt}\right)$$

$$= k \cdot \underbrace{\phantom{7000e^{kt}}}_{B}$$

② $B(0) = 7000e^{k \cdot 0} = 7000 \cdot 1 = 7000$ ✓

So $B(t) = 7000e^{kt}$ is explicit sol'n to the DE

Example 6: Verify that $y = \frac{1}{x}$ is an explicit solution of the differential equation $xy' + y = 0$.

① $\frac{dy}{dx} = y' = -\frac{1}{x^2}$ & sub into de

LHS $= \cancel{x} \cdot \left(\frac{-1}{x^{\cancel{2}}}\right) + \frac{1}{x} = -\frac{1}{x} + \frac{1}{x} = 0 =$ RHS ✓

So $y = \frac{1}{x}$ is an explicit sol'n.

Example 8: Verify that $x^2 + 4y^2 - 4 = 0$ is an implicit solution of the differential equation $\frac{dy}{dx} = -\frac{x}{4y}$. Find the interval of definition I

check w/ implicit diff:

$$\cancel{2x} + 8y\frac{dy}{dx} - 0 = 0 - 2x$$
$$-\cancel{2x}$$

$$\frac{\cancel{8y}\frac{dy}{dx}}{\cancel{8y}} = \frac{-\cancel{2}x}{\cancel{8}_4 y} \Rightarrow \frac{dy}{dx} = -\frac{x}{4y} = \text{RHS} \checkmark$$

So $x^2 + 4y^2 - 4 = 0$ is an implicit sol'n.