Mat 161 Agenda Day 25 03/22/06

Worksheet

3.4 Fundamental Theorem of Algebra
Linear Factorization Theorem
Conjugate Pairs
Factoring a Polynomial

Homework: 3.4

Quiz on
Synthetic Division,
Rational Root Theorem,
Polynomial given roots
The Fundamental Theorem of Algebra

If $f(x)$ is a polynomial of degree $n$, where $n>0$, then $f$ has at least one zero in the complex number system.

$x + 4 = 0$

$x^2 + 4 = 0$

$(x+2i)(x-2i)=0$

$x^2 - 2ix + 2x + 4 = 0$

$x = 2i$, $x = -2i$
$x^2 - 4 = (x + 2)(x - 2)$

$x^2 - 2 = (x + \sqrt{2})(x - \sqrt{2})$

$\sqrt{2}$

$x^2 + 2 = (x - \sqrt{2}i)(x + \sqrt{2}i)$
**Linear Factorization Theorem**

If $f(x)$ is a polynomial of degree $n$, where $n > 0$, $f$ has precisely $n$ linear factors

$$f(x) = a_n(x - c_1)(x - c_2)\ldots(x - c_n)$$

where $c_1, c_2, \ldots, c_n$ are complex numbers. (Zeros may not be distinct!)

$$x^2 - 6x + 9 = (x - 3)^2 = 0$$
Solve $x^3 + 6x - 7 = 0$.

$x^3 + 6x - 7 = (x^2 + x + ?)(x - 1)$

$\left\{ \begin{array}{l} \pm 1 \pm 7 \\ \pm 1 \end{array} \right\}$

$\left( \begin{array}{c} 0 \\ \pm 1 \\ 1 \end{array} \right)$

$1 \mid 0 \ 6 \ -7$

$-1 \mid 1 \ 1 \ 7 \ 0$

$-1 \mid 1 \ 0 \ \times$

$1 \ 0 \ 7$

$x = 1$

$x - 1 = 0$
\[ x^3 + 6x - 7 = 0 \quad (x-1)(x^2+x+7) \]

\[ x^2 + x + 7 = 0 \]

\[ a = 1 \quad b = 1 \quad c = 7 \]

\[ x = -1 \pm \sqrt{1 - 4(1)(7)} \]

\[ = -1 \pm \sqrt{-27} \]

\[ = -1 \pm 3\sqrt{3}i \]

\[ = \frac{-1 \pm 3\sqrt{3}i}{2} \]
$p: \pm 1, \pm 7$
$q: \pm 1$
$p/q: \pm 1, \pm 7$

Use synthetic division to find a number from the list that is a solution.

\[\begin{array}{c|cccc}
1 & 1 & 0 & 6 & -7 \\
\hline
 & 1 & 1 & 7 & 0 \\
\end{array}\]
We know have \((x - 1)(x^2 + x + 7) = 0\). \(x - 1 = 0 \Rightarrow x = 1\).

\[x^2 + x + 7 = 0 \Rightarrow x = -\frac{1}{2} \pm \frac{3\sqrt{3}}{2}i.\]
Examples: Find all zeros and write as the product of linear factors.

1. \( f(y) = y^4 - 625 \)

\[ (y+1)(y^2 + 10y + 29) \]

2. \( f(x) = x^3 + 11x^2 + 39x + 29 \)

\[ x \Rightarrow \pm 1 \pm 29 \]

\[ g \Rightarrow \pm 1 \]

\[ \frac{a}{b} \Rightarrow \pm 1 \pm 29 \]
\[ x^3 + 11x^2 + 39 + 29 = (x+1)(x+5-2i)(x+5+2i) \]

\[ x^2 + 10x + 29 \]

\[ a = 1 \]
\[ b = 10 \]
\[ c = 29 \]

\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

\[ x = \frac{-10 \pm \sqrt{100 - 4(1)(29)}}{2} \]

\[ x = \frac{-10 \pm \sqrt{-16}}{2} \]

\[ x = \frac{-10 \pm 4i}{2} \]

\[ x = -5 \pm 2i \]

\[ x + 5 - 2i = 0 \]
\[ x + 5 + 2i = 0 \]
1. \[ f(y) = y^4 - 625 \]

\[
(y^2 + 25)(y^2 - 25) \]

\[
(y^2 + 25)(y - 5)/(y + 5) \text{ Real} \]

\[
(y - 5i)(y + 5i)(y - 5)(y + 5) \text{ Complex} \]
3. \( h(x) = x^4 + 6x^3 + 10x^2 + 6x + 9 \)

\[(x+3)(x^3+3x^2+x+3)\]

\( p \Rightarrow \pm 1 \pm 3 \pm 9 \)

\( q \Rightarrow \pm 1 \)

\( p \Rightarrow \pm 1 \pm 3 \pm 9 \)

\[
\begin{array}{c|cccc}
-3 & 1 & 6 & 10 & 6 & 9 \\
-3 & -3 & -9 & -3 & -9 \\
\hline
-3 & 1 & 3 & 1 & 3 & 0 \\
\hline
-3 & 0 & -3 \\
\hline
1 & 0 & 1 & 0 \\
\end{array}
\]

\((x+3)^2(x^2+1)\) \textbf{Reals}

\((x+3)^2(x-i)(x+i)\) \textbf{Complex}
Complex Zeros Occur in Conjugate Pairs

Let $f(x)$ be a polynomial function that has real coefficients.

If $a + bi$, where $b \neq 0$, is a zero of the function, the conjugate $a - bi$ is also a zero of the function.
Factors of a Polynomial

Every polynomial of degree $n > 0$ with real coefficients can be written as the product of linear and quadratic factors with real coefficients, where the quadratic factors have no real zeros.

Definition: A quadratic factor with no real zeros is said to be prime or irreducible over the reals.

\[(y + 5)(y - 5)(y^2 + 25)\]
Factor \( f(x) = x^4 - 12x^2 - 13 \)

a) as the product of factors that irreducible over the rationals.

\[ (x^2 - 13)(x^2 + 1) \quad \text{Rationals} \]

\[ (x + \sqrt{13})(x - \sqrt{13})(x^2 + 1) \quad \text{Reals} \]

\[ (x + \sqrt{13})(x - \sqrt{13})(x - i)(x + i) \quad \text{Complex} \]
b) as the product of factors that irreducible over the reals.

\[(x + \sqrt{13})(x - \sqrt{13})(x^2 + 1)\]
c) completely.

\((x + \sqrt{13})(x - \sqrt{13})(x + i)(x - i)\)
Example: Write the polynomial $f(x) = x^4 - 2x^3 - 3x^2 + 12x - 18$

(a) as the product of factors that are irreducible over the *rationals*.

(b) as the product of linear and quadratic factors that are irreducible over the *reals*.

(c) in completely factored form.
Find the function of the following graphs. The Window is
\[ X \text{ min} = -6 \quad Y \text{ min} = -8 \]
\[ X \text{ max} = 6 \quad Y \text{ max} = 8 \]
\[ X \text{ scl} = 2 \quad Y \text{ scl} = 2 \]

1. 4th degree polynomial

\[ f(x) = (x+1)(x-2)^3 \]

\[ x = -1 \quad x = 2 \]
2. 4th degree polynomial

\[ \chi = -3 \quad \chi = -1 \quad \chi = 2 \]

\[(\chi + 3)^2 (\chi + 1)(\chi - 2)\]
\[(\chi^2 + 6\chi + 9)(\chi + 1)(\chi - 2)\]
\[\chi^3 + 6\chi^2 + 9\chi \]
\[\chi^2 + 6\chi + 9 \]
\[\chi^3 + 7\chi^2 + 15\chi + 9 \]

\[\chi - 2 \]
\[\chi^4 + 7\chi^3 + 15\chi^2 + 9\chi \]
\[-2\chi^3 - 14\chi^2 - 30\chi - 18 \]
\[\chi^4 + 5\chi^3 + \chi^2 - 21\chi - 18 \]
Find the function of the following graphs. The Window is

\[ \begin{align*}
X \text{ min} & = -6 & Y \text{ min} & = -8 \\
X \text{ max} & = 6 & Y \text{ max} & = 8 \\
X \text{ scl} & = 2 & Y \text{ scl} & = 2
\end{align*} \]

3. \text{3}^{rd} \text{ degree polynomial}

\[(x + 1)(x - 2)^2\]
4. What are the right-hand and left-hand behavior of

a. \( f(x) = 5x^3 + 7x^2 - 2x + 6 \)
\[ \begin{align*}
    x &\to -\infty \quad f(x) \to -\infty \\
    x &\to +\infty \quad f(x) \to +\infty
\end{align*} \]

b. \( g(x) = -3x^4 + 6x^2 + 4x - 10 \)
\[ \begin{align*}
    x &\to -\infty \quad g(x) \to -\infty \\
    x &\to +\infty \quad g(x) \to -\infty
\end{align*} \]
5. Factor: \(5x^3 - 28x^2 - 11x - 6 = (x - 6)(5x^2 + 2x + 1)\)

\[
s\geq \pm 1 \pm 2 \pm 3 \pm 6 \quad \frac{s}{s} \Rightarrow \pm 1 \pm 2 \pm 3 \pm 6
\]

\[
g \geq \pm 1 \pm 5
\]

\[
6 \left| \begin{array}{c} 5 \quad -2 \quad 8 \quad -11 \quad -6 \\ \hline \end{array} \right.
\]

\[
\frac{30}{5} = \frac{12}{2} = \frac{6}{1}
\]
\((5x^2 + 2x + 1)\)

\[x = \frac{-2 \pm \sqrt{4 - 4(5)(1)}}{10} = \frac{-2 \pm \sqrt{-14}}{10} = \frac{-2 \pm 4i}{10} = \frac{-1 \pm 2i}{5}\]
-1 ± 2i

\[ x = \frac{-1 \pm 2i}{5} \]

\[ 5x = -1 \pm 2i \]

\[ 5x + 1 - 2i = 0 \]

\[ (x-6)(5x+1-2i)(5x+1+2i) \]
6. Find the zeros of the polynomial $f(x) = x^4 - 8x^2 + 72x - 65$
7. Is the zero of even or odd multiplicity?
Find a polynomial with real coefficients that has the given zeros.

-1, 6 + 5i, 6 - 5i

0, 0, 4, 1 + 2i

\[ \chi = 1 - \sqrt{2}i \]
\[ \chi = 1 + \sqrt{2}i \]

\[ \chi = 0 \]
\[ \chi = 4 \]

\[ \chi^2 (\chi - 4) (\chi - 1 + \sqrt{2}i) (\chi - 1 - \sqrt{2}i) \]

\[ \chi^2 - 2\chi + 3 \]
\[ \chi^3 - 4\chi^2 \]
0,0,4,1 +\sqrt{2} \, i
Example: Write the polynomial \( f(x) = x^4 - 2x^3 - 3x^2 + 12x - 18 \)

(a) as the product of factors that are irreducible over the **rationals**.

(b) as the product of linear and quadratic factors that are irreducible over the **reals**.

(c) in completely factored form.
4. Show that $x = -2$ is a zero (or root) of $x^3 + 2x^2 - 2x - 4$. Factor completely and find all real zeros.
Examples: Find all real zeros.

8. $h(x) = -x^3 - 9x^2 + 20x - 12$
9. \[ f(z) = 12z^3 - 4z^2 - 27z + 9 \]
Examples: Find all zeros and write as the product of linear factors.

1. $f(y) = y^4 - 625$

2. $f(x) = x^3 + 11x^2 + 39x + 29$
3. \( h(x) = x^4 + 6x^3 + 10x^2 + 6x + 9 \)