Mat 161 Day 8 Agenda

Return Quiz

Chapter 1.7
Inverse Functions

Distance Formula
Equation of Circle
Exploring Data

Worksheet 4a
$y^2 = x + 25$
\[ f(x) = \begin{cases} 
-2x + 1 & x < 1 \\
3x - 2 & x \geq 1 
\end{cases} \]

\[-2(-1) + 1 = 2 + 1 = 3\]
$8 + x \neq 0$  
$(-\infty, -8) \cup (-8, \infty)$

$x \neq -8$
\[ f(x) = 3x - x^2 \]
\[ \frac{f(x+h) - f(x)}{h} \]
\[ f(x+h) = 3(x+h) - (x+h)^2 \]
\[ 3x + 3h - (x^2 + 2hx + h^2) \]
\[ f(x+h) = 3x + 3h - x^2 - 2hx - h^2 \]
\[ f(x+h) - f(x) = 3h - 2hx - h^2 \]
\[ \frac{f(x+h) - f(x)}{h} = \frac{3h - 2hx - h^2}{h} = 3 - 2x - h \quad h \neq 0 \]
\[ 5x^2 + x - x^2 = 5 - x \]
\[ 5x + x^3 - 5 - x \]
\[ (5 + x)(x - 1) = \]
\[ (5 + x)(x + 1)(x - 1) = 0 \]
\[ x = -5, -1, 1 \]
\[
\frac{12 - 4x}{x - 3} = \frac{4(3 - x)}{x - 3} = \frac{-4(x - 3)}{(x - 3)} = -4
\]

\[x \neq 3\]
Mat 161: Composition of Functions

Find

a) \((f \circ g)(x)\)

\[
f \circ g(x) = f(x+3) = \frac{1}{x+3}
\]

\(x \neq -3\)

b) \((g \circ f)(x)\)

\[
g \circ f(x) = g\left(\frac{1}{x}\right) = \frac{1}{x} + \frac{3x}{x} = \frac{1 + 3x}{x}
\]

\(x \neq 0\)

c) Domain of each

Exercise 3:

\[
f(x) = \frac{1}{x}
\]

\(D: \ x \neq 0\)

\[
g(x) = x + 3
\]

\(D: \) Reals

\(\)
Example: The weekly cost $C$ to manufacture a certain product is given by $C(x) = 25x + 5000$. The number of units $x$ produced in $t$ hours is given by $x(t) = 3t$.

(a) Find and interpret $(C \circ x)(t)$.

$$(C \circ x)(t) = C(3t) = 25(3t) + 5000$$

$NC(t) = 75t + 5000$

$10,000 = 75t + 5000$
(a) Find and interpret \((C \circ x)(t)\).

\((C \circ x)(t) = 75t + 5000\). This equation represents the cost after \(t\) production hours.
(b) After how many hours will the weekly cost be equal to $10,000?

\[ 10,000 = 75t + 5000 \]
\[ 5000 = 75t \]
\[ 66.66 \approx t \]

After approximately 66.6 hours, the cost will equal $10,000.
\[ f(x) = x - 2 \]

\[ g(x) = x + 2 \]

\[ f \circ g (x) = ? \]

\[ g \circ f (x) = ? \]

\[ (x, y) \]

\[ (x, y) \]

\[ x | y \]

\[ -1 \] \[ -3 \]

\[ 0 \] \[ -2 \]

\[ -1 \] \[ -1 \]

\[ 2 \] \[ 0 \]

\[ 3 \] \[ 1 \]
\[ f(x) = x - 2 \]

\[ g(x) = x + 2 \]

\[ f \circ g(x) = f(x+2) = x + 2 - 2 = x \]

\[ g \circ f(x) = g(x-2) = x - 2 + 2 = x \]
\[ y = x^2 \]
\[ f(x) = x^3 \]

1. \[ y = x^3 \]
   Let \( y = x \)

2. \[ x = y^\frac{1}{3} \]

3. \[ \sqrt[3]{x} = y = f^{-1}(x) \]

\[ x^{-1} = \frac{1}{x} \]
\[ x^{-4} = \frac{1}{x^4} \]
\( f(x) = x^3 \) \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad 

\( f^{-1}(x) = x^{\frac{1}{3}} \)

\[
\begin{align*}
(f \circ f^{-1})(x) &= f(x^{\frac{1}{3}}) = (x^{\frac{1}{3}})^3 = x
\end{align*}
\]
Definition: Let $f$ and $g$ be two functions such that

$f(g(x)) = x$ for every $x$ in the domain of $g$, and
$g(f(x)) = x$ for every $x$ in the domain of $f$.

Under these conditions, the function $g$ is the inverse function of the function $f$.

The function $g$ is denoted $f^{-1}$ (read “$f$-inverse”).

Therefore

$$f(f^{-1}(x)) = x \text{ and } f^{-1}(f(x)) = x.$$
The domain of $f$ must be equal to the range of $f^{-1}$ and
the range of $f$ must be equal to the domain of $f^{-1}$. 
Are these inverses of each other? 
Prove algebraically, verify graphically

\[ f(x) = \sqrt[3]{x-1} \text{ and } g(x) = x^3 + 1 \]

\[ f \circ g (x) = f(x^3 + 1) = \sqrt[3]{x^3 + 1 - 1} = \sqrt[3]{x^3} = x \]

\[ g \circ f (x) = g[(x-1)^{\frac{1}{3}}] = [(x-1)^{\frac{1}{3}}] + 1 = x - 1 + 1 = x \]
(Note that this is a Square screen.)

The graphs of $f$ and $g$ are symmetric in the line $y = x$. This is true for any pair of inverse functions.
Examples: Using common sense, find the inverses of the following functions:

\[ f(x) = x + 2 \]
\[ f(x) = \frac{x}{3} \]
\[ g(x) = x^2 \]

1. Let \( y = \frac{x}{3} \)
2. Solve for \( y \)
   \[ 3x = y \]
$g(x) = x^2 \quad x \geq 0$

$y = x^2$

$x = y^2$

$\sqrt{x} = f^{-1}(x)$

$f'(x)$
The diagram illustrates the concept of function and its inverse. The function $f(x) = x + 4$ is defined with its domain and range labeled as $x$ and $f(x)$ respectively. The inverse function $f^{-1}(x) = x - 4$ is also shown, with its domain and range labeled as $f^{-1}(x)$ and the range of $f$ respectively.
Definition: A function $f$ is one-to-one if, for $a$ and $b$ in its domain, $f(a) = f(b)$ implies that $a = b$.

A function $f$ has an inverse function $f^{-1}$ if and only if $f$ is one-to-one.
Horizontal Line Test: If every horizontal line intersects the graph of a function $f$ at most once, then the function is one-to-one. (That is, no horizontal line intersects the graph of the function more than once.)
Examples:

One-to-one function  Function, not one-to-one  Not a function

\[ \sqrt{25 - x^2} \quad x \geq 0 \]
Example: \( f(x) = \sqrt[3]{x - 5} \) and solve for \( y \).

4. Replace \( y \) by \( f^{-1} \) in the new equation.

5. Verify that \( f \) and \( f^{-1} \) are inverse functions by showing that the domain of \( f \) is equal to the range of \( f^{-1} \) and the range of \( f \) is equal to the domain of \( f^{-1} \) and that \( f(f^{-1}(x)) = x \) and \( f^{-1}(f(x)) = x \).
\[ f(x) = \sqrt[3]{x - 5} \]
\[ y = \sqrt[3]{x - 5} \]
\[ x = \sqrt[3]{y - 5} \]
\[ x^3 = y - 5 \]
\[ y = x^3 + 5 \]
\[ f^{-1}(x) = x^3 + 5 \]
Example: \( g(x) = \frac{x-4}{x+2} \)

\[ y = \frac{x-4}{x+2} \]

Let \( y = x \)

\[ x = \frac{(y-4)}{(y+2)} \]

\[ x(y+2) = y-4 \]

\[ xy + 2x = y - 4 \]

\[ -2x \quad -2x \]

\[ xy - y = y - 4 - 2x \]

\[ -y \quad -y \]

\[ xy - y = -4 - 2x \]

\[ y'(x) = \frac{-4 - 2x}{x - 1} \]
\[ f^{-1}(x) = \frac{-4-2x}{x-1} \]

\[ f(f^{-1}(x)) = f \left( \frac{-4-2x}{x-1} \right) = \left( \frac{-4-2x}{x-1} - \frac{4}{x-1} \right)^{x-1} \]

\[ = \frac{-4-2x - 4(x-1)}{-4-2x + 2(x-1)} \cdot \frac{-4-2x - 4(x-1)}{-4-2x + 2(x-1)} \]

\[ = \frac{-6x}{-6} = x \]
\[ g(x) = \frac{x - 4}{x + 2} \]
\[ y = \frac{x - 4}{x + 2} \]
\[ x = \frac{y - 4}{y + 2} \]
\[ xy + 2x = y - 4 \]
\[ xy - y = -2x - 4 \]
\[ (x - 1)y = -2x - 4 \]
\[ y = \frac{-2x - 4}{x - 1} \]
Examples: In each case, determine if the given function is one-to-one. If so, find its inverse. (From p. 153)

44. \( f(x) = 3x + 5 \)

46. \( h(x) = \frac{4}{x^2} \)

\[
y = 3x + 5
\]

\[
f(x) = 3y + 5
\]

\[
x - 5 = 3y
\]

\[
\frac{x - 5}{3} = y
\]

\[
f^{-1}(x) = \frac{x - 5}{3}
\]
48. \( q(x) = (x - 5)^2, x \leq 5 \)

\[
y = (x - 5)^2
\]

\[
y = x^2 - 10x + 25
\]
Definition of Inverse Function

Let $f$ and $g$ be two functions such that

$$f(g(x)) = x \quad \text{for every } x \text{ in the domain of } g$$

and

$$g(f(x)) = x \quad \text{for every } x \text{ in the domain of } f.$$

Under these conditions, the function $g$ is the inverse function of the function $f$. The function $g$ is denoted by $f^{-1}$ (read “$f$-inverse”). So,

$$f(f^{-1}(x)) = x \quad \text{and} \quad f^{-1}(f(x)) = x.$$

The domain of $f$ must be equal to the range of $f^{-1}$, and the range of $f$ must be equal to the domain of $f^{-1}$. 
Finding an Inverse Function

1. Use the Horizontal Line Test to decide whether $f$ has an inverse function.
2. In the equation for $f(x)$, replace $f(x)$ by $y$.
3. Interchange the roles of $x$ and $y$, and solve for $y$.
4. Replace $y$ by $f^{-1}(x)$ in the new equation.
5. Verify that $f$ and $f^{-1}$ are inverse functions of each other by showing that the domain of $f$ is equal to the range of $f^{-1}$, the range of $f$ is equal to the domain of $f^{-1}$, and $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$. 
1. Identify the common parent function and the transformation shown in the graph:

a) parent function:

b) function for the graphed (transformed function)
2. Find the standard form of the circle with center (-3, 2) and radius 5.
3. Find the distance between the points \((-3, 2)\) and \((7, -5)\)
The distance $d$ between the two points is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$
4. Factor: \((x^2 + 1)^{-2} - x^2(x^2 + 1)^{-3}\)
5. Use the table below to find the percent increase of radio stations with Jazz format.

<table>
<thead>
<tr>
<th>Year</th>
<th>Number of Radio Stations with Jazz Format</th>
</tr>
</thead>
<tbody>
<tr>
<td>1999</td>
<td>243</td>
</tr>
<tr>
<td>2001</td>
<td>213</td>
</tr>
</tbody>
</table>
Page 55 #22. The table shows the number $y$ of Wal-Mart stores for each year $x$ from 1994 through 2001. Sketch a scatter plot of the data.

<table>
<thead>
<tr>
<th>Year, $x$</th>
<th>Number of stores, $y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1994</td>
<td>2759</td>
</tr>
<tr>
<td>1995</td>
<td>2943</td>
</tr>
<tr>
<td>1996</td>
<td>3054</td>
</tr>
<tr>
<td>1997</td>
<td>3406</td>
</tr>
<tr>
<td>1998</td>
<td>3599</td>
</tr>
<tr>
<td>1999</td>
<td>3985</td>
</tr>
<tr>
<td>2000</td>
<td>4189</td>
</tr>
<tr>
<td>2001</td>
<td>4414</td>
</tr>
</tbody>
</table>
A student's salary in dollars for collecting phone books is given by $S(x) = 30x + 240$

The amount of withholding for taxes is $W(x) = 0.20x$, where $x$ is salary. Express withholding as a function of number of phone books collected.
A company produces a toy for which the variable cost is $12.30 per unit and the fixed costs are $98,000. The toy sells for $17.98. Let $x$ be the number of units produced and sold.
An open box of maximum volume is to be made from a square piece of material 24 centimeters on a side, by cutting out equal squares from the corners and turning up the sides.

Find the function for Volume of the box.
Distance Formula, Midpoint, and Equation of a Circle

Given two points in the coordinate plane: $(x_1, y_1), (x_2, y_2)$

The **distance** $d$ between the two points is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$
Find the distance between (2, -5) and (8, 3).

\[ d = \sqrt{(8 - 2)^2 + (3 - (-5))^2} \]
\[ = \sqrt{6^2 + 8^2} \]
\[ = \sqrt{36 + 64} \]
\[ = \sqrt{100} \]
\[ = 10 \]
The midpoint of the line segment joining the two points is
\[
\left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right).
\]

Find the midpoint of the line segment joining the points

\((-9, 5)\) and \((4, 2)\).

\[
\left( \frac{-9 + 4}{2}, \frac{5 + 2}{2} \right) = \left( \frac{-5}{2}, \frac{7}{2} \right)
\]
Find the standard form of the equation of the circle with center at (2, -5) and radius 4.

\[(x - 2)^2 + (y - (-5))^2 = 4^2\]

or

\[(x - 2)^2 + (y + 5)^2 = 16\]
Examples: Factor completely.

\[ x^2 - x - 6 \]

\[ 2x^2 - 3x - 14 \]

\[ 6x^2 - 29x + 35 \]

To see a review of quadratics factoring, go to the following site:

http://faculty.mc3.edu/rhofman/Flash03/Dec17/menufactor.html
P.4 Rational Expressions

Definitions:
The set of real numbers for which an expression is defined is its **domain**. Two expressions are **equivalent** if they yield the same value for all numbers in their domain.

The quotient of two algebraic expressions is a **fractional expression**.
The quotient of two polynomials is a **rational expression**.
Domains:
For a polynomial, the domain is all real numbers.
For expressions that contain radicals with even indices, the radicand must be greater than or equal to zero.
For rational expressions, the denominator cannot be equal to zero.

Examples:

$5x^3 - 2x^2 + 6x - 8$ is a polynomial, the domain is all real numbers.

$\sqrt{x-2}$, $x - 2 \geq 0$; the domain is $[2, \infty)$.

$\frac{x}{x-1}$, $x - 1 \neq 0$, the domain is $(-\infty, 1) \cup (1, \infty)$. 
Find the domain:

\[ 2x^2 - 5x - 2 \]

\[ 6x^2 - 9, x > 0 \]

\[ \frac{x + 1}{2x + 1} \]

\[ \sqrt{6 - x} \]
Simplifying Rational Expressions

The key to simplifying rational expressions is that the numerator and denominator should not have any common factors. If the numerator and denominator do have common factors, they can be eliminated by using

\[
\frac{ac}{bc} = \frac{a}{b} \cdot \frac{c}{c} = \frac{a}{b} \cdot 1 = \frac{a}{b}
\]

Note that \( \frac{c}{c} = 1 \) only if \( c \neq 0 \).
Reduce: \[ \frac{x^2 - 2x + 1}{x^2 - 1} \]
\[
\frac{x^2 - 2x + 1}{x^2 - 1} = \frac{(x - 1)^2}{(x + 1)(x - 1)} \\
= \frac{x - 1}{x + 1} \cdot \frac{x - 1}{x - 1} \\
= \frac{x - 1}{x + 1}, \quad x \neq 1
\]
Reduce:

\[ \frac{x^2 + 8x - 20}{x^2 + 11x + 10} \]

\[ \frac{x^2 - 9}{x^3 + x^2 - 9x - 9} \]
Simplify: \( \frac{x^3 + 1}{x^2 - 6x + 9} \cdot \frac{x^2 - 9}{x^2 - x + 1} \)
\[
\frac{x^2 + 1}{x^2 - 6x + 9} \cdot \frac{x^2 - 9}{x^2 - x + 1} \\
= \frac{(x+1)(x^2 - x + 1)}{(x - 3)^2} \cdot \frac{(x+3)(x - 3)}{x^2 - x + 1} \\
= \frac{(x+1)(x^2 - x + 1)(x+3)(x - 3)}{(x - 3)^2(x^2 - x + 1)} \\
= \frac{(x+1)(x+3)}{(x-3)} \cdot \frac{|x^2 - x + 1|(x - 3)}{|x^2 - x + 1|(x - 3)} \\
= \frac{(x+1)(x+3)}{(x-3)} \cdot 1, x \neq 3 \\
= \frac{(x+1)(x+3)}{(x-3)}, x \neq 3
\]
Simplify: \[ \frac{x+6}{x} + \frac{2}{x^2 + x} \]
\[
\frac{x + 6}{x} + \frac{2}{x^2 + x} = \frac{x + 6}{x} \cdot \frac{x + 1}{x + 1} + \frac{2}{x(x + 1)}
\]

\[
= \frac{(x^2 + 7x + 6) + 2}{x(x + 1)}
\]

\[
= \frac{x^2 + 7x + 8}{x(x + 1)}
\]
Complex Fractions:
Fractions that contain fractions in the numerator or denominator are called complex fractions. There are two ways to simplify a complex fraction. Here are examples of each.
Order of Operations:

\[
1 - \frac{2}{x} \cdot \frac{x - 2}{x} = \frac{x}{9x - 4} \cdot \frac{x}{x(x - 1)}
\]

\[
= \frac{x - 2}{x} \cdot \frac{x(x - 1)}{9x - 4}
\]

\[
= \frac{(x - 2)(x - 1)}{9x - 4}, \quad x \neq 0
\]
Multiply by the LCD.

\[
\frac{1 - \frac{2}{x}}{\frac{5}{x-1} + \frac{4}{x}} = \frac{1 - \frac{2}{x}}{\frac{5}{x-1} + \frac{4}{x}} \cdot \frac{x(x-1)}{x(x-1)}
\]

\[
= \frac{x(x-1) - 2(x-1)}{5x + 4(x-1)}
\]

\[
= \frac{x^2 - 3x + 2}{9x - 4}
\]
Rationalize the numerator: \( \frac{\sqrt{x + 1} - 1}{x} \)
\[
\frac{\sqrt{x+1}-1}{x} = \frac{\sqrt{x+1}-1}{x} \cdot \frac{\sqrt{x+1}+1}{\sqrt{x+1}+1}
\]

\[
= \frac{(\sqrt{x+1})^2 - 1^2}{x(\sqrt{x+1}+1)}
\]

\[
= \frac{x}{x(\sqrt{x+1}+1)}
\]

\[
= \frac{1}{\sqrt{x+1}+1}, \; x \neq 0
\]
Perform the operations and simplify:

\[
\frac{4y - 16}{5y + 15} \div \frac{2y + 6}{4 - y}
\]
\[
\frac{2}{x^2 - x - 2} + \frac{10}{x^2 + 2x - 8}
\]
\[
\frac{2}{x+1} + \frac{2}{x-1} + \frac{1}{x^2 - 1}
\]
\[
\frac{x - 4}{\frac{x}{4} - \frac{4}{x}}
\]
\[ 5x^5 - 3x^{-\frac{3}{2}} \]
\[2x(x - 5)^{-3} - 4x^2(x - 5)^{-4}\]