Chapter 1.6

Combinations of Functions
Distance Formula
Equation of Circle

Quiz 2
1.6 Combinations of Functions
- Arithmetic Combinations of Functions
- Composition of Functions
P.5 The Cartesian Plane

- The Cartesian Plane
- Distance Formula
- Midpoint Formula
- Equation of a Circle
Graphs of Common Functions

Constant Function  Identity Function  Absolute Value Function

Square Root Function  Quadratic Function  Cubic Function
Let $f$ and $g$ be two functions with overlapping domains. Then, for all $x$ common to both domains, the sum, difference, product, and quotient of $f$ and $g$ are defined as follows.

**Sum:** \((f + g)(x) = f(x) + g(x)\)

**Difference:** \((f - g)(x) = f(x) - g(x)\)

**Product:** \((fg)(x) = f(x) \cdot g(x)\)

**Quotient:** \(\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \quad g(x) \neq 0\)
Examples:
\( f(x) = x^2 + 2x \) and \( g(x) = 2x + 1 \). Find the following.

a) \((f + g)(x) = f(x) + g(x)\)
\[
(f + g)(x) = x^2 + 2x + 2x + 1 = x^2 + 4x + 1
\]

b) \((f - g)(x) = f(x) - g(x)\)
\[
(f - g)(x) = (x^2 + 2x) - (2x + 1) = x^2 + 2x - 2x - 1 = x^2 - 1
\]

c) \((f \cdot g)(x) = f(x) \cdot g(x)\)
\[
(f \cdot g)(x) = x^2 \cdot 2x + 1 = 2x^3 + 1
\]
\[
2x^3 + 4x^2
\]
\[
\frac{x^2 + 2x}{2x^3 + 5x^2 + 2x}
\]
Examples.
\[ f(x) = x^2 + 2x \text{ and } g(x) = 2x + 1. \]
\[ \left( \frac{f}{g} \right)(x) = \frac{x^2 + 2x}{2x + 1} = \frac{x(x + 2)}{2x + 1}, \quad x \neq -\frac{1}{2} \]
\( f(x) = x^2 + 2x \) and \( g(x) = 2x + 1 \). Find the following.

a) 
\[(f + g)(x) = f(x) + g(x)\]
\[= (x^2 + 2x) + (2x + 1)\]
\[= x^2 + 4x + 1\]
\( (f - g)(x) = f(x) - g(x) \)
b) 
\[(f - g)(x) = f(x) - g(x)\]
\[= \left( x^2 + 2x \right) - (2x + 1) \]
\[= x^2 - 1 \]
c) \((fg)(x) = f(x) \cdot g(x)\)
\[(f \cdot g)(x) = f(x) \cdot g(x)\]
\[= (x^2 + 2x)(2x + 1)\]
\[= 2x^3 + 5x^2 + 2x\]
\[
\left( \frac{f}{g} \right)(x) = \frac{f(x)}{g(x)}
\]
d)\[
\left( \frac{f}{g} \right)(x) = \frac{f(x)}{g(x)}
\]
\[
= \frac{x^2 + 2x}{2x + 1}
\]
Example: Given \( f(x) = \frac{x}{x+1}, g(x) = x^3 \),

find \( f + g, f - g, fg, \frac{f}{g} \).

\( f(x) \rightarrow \text{domain } x \neq -1 \)

\( g(x) \rightarrow \text{all reals } x \neq -1 \)

\[
(f + g)(x) = \frac{x}{x+1} + \frac{x^3}{x+1} = \frac{x + x + x^3}{x+1} = \frac{x^4}{x+1} \]

\[
(f - g)(x) = \frac{x}{x+1} - \frac{x^3}{x+1} = \frac{x - x^4 - x^3}{x+1} \]

\[
(f \cdot g)(x) = \left( \frac{x}{x+1} \right) \cdot \frac{x^3}{1} = \frac{x^4}{x+1} \]

\[
\left( \frac{f}{g} \right)(x) = \frac{x}{x+1} \cdot \frac{x^3}{x^3} = \frac{x}{x+1} \cdot \frac{x^3}{x^3} = \frac{1}{x+1} \]

\( x \neq -1, 0, x^2 \)
Domain of $f + g$ is restricted to the overlap of the domains of $f$ and $g$. 
Composition of Functions
Definition: The composition of the function $f$ with the function $g$ is $(f \circ g)(x) = f(g(x))$.
The domain of $f \circ g$ is the set of all $x$ in the domain of $g$ such that $g(x)$ is in the domain of $f$.

$$f \circ g(x) = f(g(x))$$
\( f(x) = x^2 + 2x \) and \( g(x) = 2x + 1 \). Find the following.

a) \((f \circ g)(x) = f(g(x))\)

\[
(f \circ g)(x) = f(g(x)) = f(2x+1) = (2x+1)^2 + 2(2x+1) = 4x^2 + 4x + 1 + 4x + 2 = 4x^2 + 8x + 3
\]

\[
g \circ f(x) = g(f(x)) = g(x^2 + 2x) = 2(x^2 + 2x) + 1 = 2x^2 + 4x + 1
\]
$f(x) = x^2 + 2x$ and $g(x) = 2x + 1$. Find the following.

a) 

$(f \circ g)(x) = f(g(x))$

$= f(2x + 1)$

$= (2x + 1)^2 + 2(2x + 1)$

$= 4x^2 + 4x + 1 + 4x + 2$

$= 4x^2 + 8x + 3$
b) 
\( (g \circ f)(x) = g(f(x)) \)
b) 
\[(g \circ f)(x) = g(f(x))\]
\[= g(x^2 + 2x)\]
\[= 2(x^2 + 2x) + 1\]
\[= 2x^2 + 4x + 1\]
Functions: \( f(x) = x^2 + 1 \) and \( g(x) = 2x - 5 \)

\[ f \circ g \]

\[ f \cdot g \]

Range of \( g \)

Domain of \( g \)

Domain of \( f \)

\[ g(3) = 1 \]

\[ f(1) = 1 + 1 = 2 \]

\[ f(g(3)) = f(1) = 2 \]

\[ g(3) = 2(3) - 5 = 1 \]

\[ f(g(x)) = f(2x - 5) \]

\[ g'(x) = 4x^2 - 20x + 25 + 1 \]

\[ g'(3) = 36 - 60 + 26 = 2 \]
Mat 161: Composition of Functions

Find
a) \((f \circ g)(x)\)
b) \((g \circ f)(x)\)
c) Domain of each
Exercise 1:

\[ f(x) = \sqrt{x} \quad x \geq 0 \]

\[ g(x) = x - 1 \quad x \geq 1 \]

Find:

\[ (f \circ g)(2) \]

\[ (f \circ g)(0) \]

\[ f \circ g (x) = f(g(x)) = f(x-1) = \sqrt{x-1} \]

\[ f \circ g(0) = \sqrt{-1} \quad \text{no go!} \]

\[ f(g(2)) = f(2-1) = f(1) = 1 \]

\[ g \circ f(x) = g(\sqrt{x}) = \sqrt{x} - 1 \]

\[ x \geq 0 \Rightarrow x \geq 1 \]
Exercise 1:
\[ f(x) = \sqrt{x} \quad x \geq 0 \]
\[ g(x) = x - 1 \quad x \geq 1 \]

Find:

Diagram:

\[ \text{Domain: } x \geq 0 \]
\[ \text{Codomains: } \{0, \ldots\} \]
\[ \text{Range of } f \]
\[ \text{Range of } g \]
\[ \text{Range of } f \circ g \]
Exercise 2:

\[ f(x) = \sqrt{x + 4} \]
\[ g(x) = x^2 \]

**Domain:** all Reals  \( R: x \geq 0 \)

**Range:**  \( R: y \geq 0 \)

\[ f \circ g (x) = f(g(x)) = \sqrt{x^2 + 4} \quad \text{all Reals} \]

\[ g \circ f (x) = g(f(x)) = (\sqrt{x + 4})^2 = (x + 4)^{\frac{1}{2}} \cdot 2 = x + 4 \]

**Domain:**  \( x \geq -4 \)
5 = \sqrt{25} = \sqrt{16 + 9} \neq \sqrt{16} + \sqrt{9} = 4 + 3 = 7

\sqrt{x^2 + y^2} \neq \sqrt{x^2} + \sqrt{y^2} = x + y

\sqrt{x^2 + 4}
\sqrt{(x+2)^2} = [ (x+2)^2 ]^{\frac{1}{2}} = x + 2
Exercise 3:

\[ f(x) = \frac{1}{x} \]

\[ g(x) = x + 3 \]
Example: The weekly cost $C$ to manufacture a certain product is given by $C(x) = 25x + 5000$. The number of units $x$ produced in $t$ hours is given by $x(t) = 3t$.

(a) Find and interpret $(C \circ x)(t)$. 
(a) Find and interpret \((C \circ x)(t)\).

\((C \circ x)(t) = 75t + 5000\). This equation represents the cost after \(t\) production hours.
(b) After how many hours will the weekly cost be equal to $10,000?

10,000 = 75t + 5000
5000 = 75t
66.66 \approx t

After approximately 66.6 hours, the cost will equal $10,000.
3. \( x^2(x^2 + 1)^{-5} - (x^2 + 1)^{-4} \)
4. \[ 2(2+x)^{-1} + (1-x)(2+x)^{-3} \]
5. \[ 3(x - 2)^{-\frac{1}{3}} - x(x - 2)^{-\frac{4}{3}} \]
6. \((1-x^3)^{\frac{1}{3}} - x^3(1-x^3)^{\frac{-2}{3}}\)
Distance Formula, Midpoint, and Equation of a Circle

Given two points in the coordinate plane: \((x_1, y_1), (x_2, y_2)\)

The distance \(d\) between the two points is

\[d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.\]
Find the distance between (2, -5) and (8, 3).

\[ d = \sqrt{(8 - 2)^2 + (3 - (-5))^2} \]
\[ = \sqrt{6^2 + 8^2} \]
\[ = \sqrt{36 + 64} \]
\[ = \sqrt{100} \]
\[ = 10 \]
The **midpoint** of the line segment joining the two points is
\[
\left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right).
\]

Find the midpoint of the line segment joining the points
(-9, 5) and (4, 2).

\[
\left( \frac{-9 + 4}{2}, \frac{5 + 2}{2} \right) = \left( -\frac{5}{2}, \frac{7}{2} \right)
\]
Find the standard form of the equation of the circle with center at (2, -5) and radius 4.

\[(x - 2)^2 + (y - (-5))^2 = 4^2\]

or

\[(x - 2)^2 + (y + 5)^2 = 16\]
Examples: Factor completely.

\[ x^2 - x - 6 \]

\[ 2x^2 - 3x - 14 \]

\[ 6x^2 - 29x + 35 \]

To see a review of quadratics factoring, go to the following site:

http://faculty.mc3.edu/rhofman/Flash03/Dec17/menufactor.html
P.4 Rational Expressions

Definitions:
The set of real numbers for which an expression is defined is its **domain**. Two expressions are **equivalent** if they yield the same value for all numbers in their domain.

The quotient of two algebraic expressions is a **fractional expression**.
The quotient of two polynomials is a **rational expression**.
Domains:
For a polynomial, the domain is all real numbers.
For expressions that contain radicals with even indices, the radicand
must be greater than or equal to zero.
For rational expressions, the denominator cannot be equal to zero.

Examples:

$5x^3 - 2x^2 + 6x - 8$ is a polynomial, the domain is all real numbers.

$\sqrt{x-2}, x - 2 \geq 0$; the domain is $[2, \infty)$.

$\frac{x}{x-1}, x - 1 \neq 0$, the domain is $(-\infty, 1) \cup (1, \infty)$. 
Find the domain:

\[ 2x^2 - 5x - 2 \]

\[ 6x^2 - 9, x > 0 \]

\[ \frac{x + 1}{2x + 1} \]

\[ \sqrt{6 - x} \]
**Simplifying Rational Expressions**

The key to simplifying rational expressions is that the numerator and denominator should not have any common factors. If the numerator and denominator do have common factors, they can be eliminated by using

\[
\frac{ac}{bc} = \frac{a}{b} \cdot \frac{c}{c} = \frac{a}{b} \cdot 1 = \frac{a}{b}
\]

Note that \( \frac{c}{c} = 1 \) only if \( c \neq 0 \).
Reduce: \( \frac{x^2 - 2x + 1}{x^2 - 1} \)
\[
\frac{x^2 - 2x + 1}{x^2 - 1} = \frac{(x - 1)^2}{(x + 1)(x - 1)}
\]
\[
= \frac{x - 1}{x + 1} \cdot \frac{x - 1}{x - 1}
\]
\[
= \frac{x - 1}{x + 1}, x \neq 1
\]
Reduce:

\[
\frac{x^2 + 8x - 20}{x^2 + 11x + 10}
\]

\[
\frac{x^2 - 9}{x^3 + x^2 - 9x - 9}
\]
Simplify: \[
\frac{x^3 + 1}{x^2 - 6x + 9} \cdot \frac{x^2 - 9}{x^2 - x + 1}
\]
\[
\frac{x^2 + 1}{x^2 - 6x + 9} \cdot \frac{x^2 - 9}{x^2 - x + 1} \\
= \frac{(x + 1)(x^2 - x + 1)}{(x - 3)^2} \cdot \frac{(x + 3)(x - 3)}{x^2 - x + 1} \\
= \frac{(x + 1)(x^2 - x + 1)(x + 3)(x - 3)}{(x - 3)^2 (x^2 - x + 1)} \\
= \frac{(x + 1)(x + 3)}{(x - 3)} \cdot \frac{(x^2 - x + 1)(x - 3)}{(x^2 - x + 1)(x - 3)} \\
= \frac{(x + 1)(x + 3)}{(x - 3)}, x \neq 3 \\
= \frac{(x + 1)(x + 3)}{(x - 3)}, x \neq 3
\]
Simplify: \[ \frac{x + 6}{x} + \frac{2}{x^2 + x} \]
\[ \frac{x + 6}{x} + \frac{2}{x^2 + x} = \frac{x + 6}{x} \cdot \frac{x + 1}{x + 1} + \frac{2}{x(x + 1)} \]

\[ = \frac{(x^2 + 7x + 6) + 2}{x(x + 1)} \]

\[ = \frac{x^2 + 7x + 8}{x(x + 1)} \]
Complex Fractions:
Fractions that contain fractions in the numerator or denominator are called complex fractions. There are two ways to simplify a complex fraction. Here are examples of each.
Order of Operations:

\[
\frac{1 - \frac{2}{x}}{\frac{5}{x - 1} + 4} = \frac{x - 2}{\frac{x}{x(x - 1)}}
\]

\[
= \frac{x - 2}{x} \cdot \frac{x(x - 1)}{9x - 4}
\]

\[
= \frac{(x - 2)(x - 1)}{9x - 4} \cdot \frac{x}{x}
\]

\[
= \frac{(x - 2)(x - 1)}{9x - 4}, \quad x \neq 0
\]
Multiply by the LCD.

\[
\frac{1 - \frac{2}{x}}{\frac{5}{x-1} + \frac{4}{x}} = \frac{1 - \frac{2}{x}}{\frac{5}{x-1} + \frac{4}{x}} \cdot \frac{x(x-1)}{x(x-1)}
\]

\[
= \frac{x(x-1) - 2(x-1)}{5x + 4(x-1)}
\]

\[
= \frac{x^2 - 3x + 2}{9x - 4}
\]
Rationalize the numerator: \( \frac{\sqrt{x} + 1 - 1}{x} \)
\[
\frac{\sqrt{x+1} - 1}{x} = \frac{\sqrt{x+1} - 1}{x} \cdot \frac{\sqrt{x+1} + 1}{\sqrt{x+1} + 1}
\]

\[
= \frac{(\sqrt{x+1})^2 - 1^2}{x(\sqrt{x+1} + 1)}
\]

\[
= \frac{x}{x(\sqrt{x+1} + 1)}
\]

\[
= \frac{1}{\sqrt{x+1} + 1}, x \neq 0
\]
Perform the operations and simplify:

\[
\frac{4y - 16}{5y + 15} \cdot \frac{2y + 6}{4 - y}
\]
\[ \frac{2}{x^2 - x - 2} + \frac{10}{x^2 + 2x - 8} \]
\[
\frac{x - 4}{x} - \frac{4}{4} - \frac{x}{x}
\]