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3.4 Conjugate Pairs
   Factoring a Polynomial
3.5 Rational Functions and Asymptotes
   Rational Functions
   Horizontal and Vertical Asymptotes
   Applications

Worksheet

Homework: 3.4 & 3.5
\( f'(x) = (x-k)f(x) + r(x) \)

\[ f(x) = 0x^3 - x^2 - 14x + 11 \]

\[ f(4) = 64 - 16 - 56 + 11 = -1 \]

\[ k = 4 \]

\[ \pm 11 \pm 1 \]

\[ x^3 - x^2 - 14x + 11 = (x-4)(x^2 + 3x - 2) + 3 \]
Problem 37

\[2x^3 - 15x^2 + 27x - 10 = 0\]

\[
\begin{array}{cccc}
2 & -15 & 27 & -10 \\
2 & -7 & 10 \\
2 & -14 & 20 & 0 \\
\end{array}
\]

\[2(x - \frac{1}{2})(x^2 - 7x + 10)\]

\[2(x - \frac{1}{2})(x - 5)(x - 2)\]

\[x = \frac{1}{2}\]
\( f(x) = 2x^3 + x^2 - 5x + 2 \)

\[
\begin{array}{cccc}
-2 & 1 & -5 & 2 \\
2 & 0 & 1 & 0 \\
& 0 & 0 & 0
\end{array}
\]

\( 2x^3 + x^2 - 5x + 2 = (x+2)(x-1)(2x-1) \)
Real Zeros of Polynomial Functions

If \( f \) is a polynomial function and \( a \) is a real number, the following statements are equivalent.

1. \( x = a \) is a zero of the function \( f \).
2. \( x = a \) is a solution of the polynomial equations \( f(x) = 0 \).
3. \((x-a)\) is a factor of the polynomial \( f(x) \).
4. \((a,0)\) is an \( x \)-intercept of the graph of \( f \).
Using the Remainder in Synthetic Division

In summary, the remainder \( r \), obtained in synthetic division of \( f(x) \) by \( x - k \), provides the following information.

1. The remainder \( r \) gives the exact value of \( f \) at: \( r = f(k) \).
2. If \( r = 0, (x - k) \) is a factor of \( f(x) \).
3. If \( r = 0, (k, 0) \) is an \( x \)-intercept of the graph of \( f \).
Old Business: Seeing is not believing

7. \( f(x) = 6x^3 - x^2 - 13x + 8 \)
5. Use the Zero feature of your calculator to approximate the zeros of \( f(s) = s^3 - 12s^2 + 40s - 24 \) to three decimal places. Determine one of the exact zeros and use synthetic division to verify it. Factor completely.

\[ \frac{\pm 1 \pm 2 \pm 3 \pm 4 \pm 6 \pm 8 \pm 12 \pm 24}{(x-6)(x^2-6x+4) \rightarrow \text{rational}} \]

\[ (x-6)(x-3-\sqrt{5})(x-3+\sqrt{5}) \]

\[
\begin{array}{c|cccc}
6 & 1 & -12 & 40 & -24 \\
\hline
 & 6 & -36 & 24 & \\
\end{array}
\]

\[
x = \frac{6 \pm \sqrt{36 - 4(1)(4)}}{2} = \frac{6 \pm \sqrt{20}}{2} = 3 \pm \sqrt{5}
\]
The Fundamental Theorem of Algebra

If $f(x)$ is a polynomial of degree $n$, where $n > 0$, then $f$ has at least one zero in the complex number system.
Linear Factorization Theorem

If \( f(x) \) is a polynomial of degree \( n \), where \( n>0 \), \( f \) has precisely \( n \) linear factors

\[
f(x) = a_n (x - c_1)(x - c_2)...(x - c_n)
\]

where \( c_1, c_2, ..., c_n \) are complex numbers. (Zeros may not be distinct!)
Solve \( x^3 + 6x - 7 = 0 \).

\[
\begin{array}{c}
\pm 1 \pm 7 \\
1 | 1 0 6 -7 \\
\hline \\
1 1 7 0
\end{array}
\]

\[g: \pm 1, \pm 7\]
\[g: \pm 1\]
\[g: \pm 1, \pm 7\]

Use synthetic division to find a number from the list that is a solution.

\[
\begin{array}{c}
-1 | 1 0 6 -7 \\
\hline \\
1 1 7 0
\end{array}
\]

\[x^3 + 6x - 7 = (x-1)(x^2 + x + 7)\]

\[x - 1 + \sqrt{27} (x-1) \quad \text{or} \quad x + 1 - 3i \cdot \sqrt{2} \]

\[-1 + \sqrt{1 - 4(0)(7)} \quad \text{or} \quad x + 1 - 3i \cdot \sqrt{3} \]

\[-1 + \sqrt{27} \quad \text{or} \quad \frac{-1 \pm \sqrt{27}}{2} \]

\[-1 + 2i \cdot \sqrt{3} \quad \text{or} \quad \frac{-1 \pm 3i \cdot \sqrt{3}}{2} \]
Examples: Find all zeros and write as the product of linear factors.

1. \( f(y) = y^4 - 625 \)

2. \( f(x) = x^3 + 11x^2 + 39x + 29 \)
3. \[ h(x) = x^4 + 6x^3 + 10x^2 + 6x + 9 = (x+3)^2 \left( x^2 + 1 \right) \]

\[
\begin{array}{cccc}
-3 & | & 1 & 6 & 10 & 6 & 9 \\
   &   & -3 & -9 & -3 & -9 \\
-3 & | & 1 & 3 & 1 & 3 & 0 \\
   &   & -3 & 0 & -3 \\
\end{array}
\]

\[
\begin{array}{cccc}
-3 & | & 1 & 0 & 1 & 0 \\
\end{array}
\]
Complex Zeros Occur in Conjugate Pairs

Let $f(x)$ be a polynomial function that has real coefficients.

If $a + bi$, where $b \neq 0$, is a zero of the function, the conjugate $a - bi$ is also a zero of the function.
Factors of a Polynomial

Every polynomial of degree $n > 0$ with real coefficients can be written as the product of linear and quadratic factors with real coefficients, where the quadratic factors have no real zeros.

Definition: A quadratic factor with no real zeros is said to be prime or irreducible over the reals.
Factor $f(x) = x^4 - 12x^2 - 13$

a) as the product of factors that irreducible over the rationals.

$$(x^2 - 13)(x^2 + 1)$$
b) as the product of factors that are irreducible over the reals.

\[(x + \sqrt{13})(x - \sqrt{13})(x^2 + 1)\]
c) completely.

\[(x + \sqrt{13})(x - \sqrt{13})(x + i)(x - i)\]
Example: Write the polynomial \( f(x) = x^4 - 2x^3 - 3x^2 + 12x - 18 \)

(a) as the product of factors that are irreducible over the \textit{rationals},

(b) as the product of linear and quadratic factors that are irreducible over the \textit{reals},

(c) in completely factored form.
Definition: A rational function is one that can be written in the form

$$f(x) = \frac{N(x)}{D(x)}$$

where $N(x)$ and $D(x)$ are polynomials and $D(x)$ is not the zero polynomial.
Examples:

\[ f(x) = \frac{3x}{x-2} \]

Continuous mode  Dot mode
$g(x) = \frac{1}{x^2 - 9}$

Zoom Standard  
Zoom Decimal
\[ h(x) = \frac{x^2 - 4}{x + 2} \]
Definitions:
The line \( x = a \) is a **vertical asymptote** of the graph of \( f \) if \( f(x) \to \infty \) or \( f(x) \to -\infty \) as \( x \to a \), either from the right or from the left.

The line \( y = b \) is a **horizontal asymptote** of the graph of \( f \) if \( f(x) \to b \) as \( x \to \infty \) or \( x \to -\infty \).
Asymptotes of a Rational Function

Let \( f \) be the rational function

\[
f(x) = \frac{N(x)}{D(x)} = \frac{a_n x^n + \ldots + a_1 x + a_0}{b_m x^m + \ldots + b_1 x + b_0}
\]

where \( N(x) \) and \( D(x) \) have no common factors.

1. The graph of \( f \) has vertical asymptotes at the zeros of \( D(x) \).
2. The graph of \( f \) has at most one horizontal asymptote determined by comparing the degrees of \( N(x) \) and \( D(x) \).
   a. If \( n < m \), the graph of \( f \) has the line \( y = 0 \) (the x-axis) as a horizontal asymptote.
   b. If \( n = m \), the graph of \( f \) has the line \( y = \frac{a_n}{b_m} \) as a horizontal asymptote.
   c. If \( n > m \), the graph of \( f \) has no horizontal asymptote.
Asymptotes of a Rational Function

Let \( f \) be the rational function

\[
f(x) = \frac{N(x)}{D(x)} = \frac{a_n x^n + \ldots + a_1 x + a_0}{b_m x^m + \ldots + b_1 x + b_0}
\]

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1. The graph of \( f \) has vertical asymptotes at the zeros of \( D(x) \).
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   a. If \( n < m \), the graph of \( f \) has the line \( y = 0 \) (the x-axis) as a horizontal asymptote.
   b. If \( n = m \), the graph of \( f \) has the line \( y = \frac{a_n}{b_m} \) as a horizontal asymptote.
   c. If \( n > m \), the graph of \( f \) has no horizontal asymptote.
Guidelines for Graphing Rational Functions

Let \( f(x) = \frac{N(x)}{D(x)} \), where \( N(x) \) and \( D(x) \) are polynomials.

1. Simplify \( f \), if possible.
2. Find and plot the \( y \)-intercept (if any) by evaluating \( f(0) \).
3. Find the zeros of the numerator (if any) by solving the equation \( N(x) = 0 \). Then plot the corresponding \( x \)-intercepts.
4. Find the zeros of the denominator (if any) by solving the equation \( D(x) = 0 \). Then sketch the corresponding vertical asymptotes using dashed vertical lines.
5. Find and sketch the horizontal asymptote (if any) of the graph using a dashed horizontal line.
6. Plot at least one point \( \text{between} \) and one point \( \text{beyond} \) each \( x \)-intercept and vertical asymptote.
7. Use smooth curves to complete the graph between and beyond the vertical asymptotes.
\[ f(x) = \frac{x + 1}{x^2 - 1} \]

Horizontal asymptote is \( y = 0 \). The only vertical asymptote is \( x = 1 \). There will be a hole in the graph at \( x = -1 \).
\[ g(x) = \frac{2x + 5}{4x - 6}. \]

The horizontal asymptote is at \( y = 1/2 \), and the vertical asymptote is at \( x = 3/2 \).
$$h(x) = \frac{x^2}{x + 1}.$$ 

No horizontal asymptote and a vertical asymptote at $x = -1$.
A game commission has determined that if 500 deer are introduced into a preserve, the population at any time $t$ (in months) is given by

$$N = \frac{500 + 350t}{1 + 0.2t}.$$  What is the carrying capacity of the preserve?

The carrying capacity will be the horizontal asymptote, $y = 1750$. 
Examples: (a) Find the domain of the function, (b) identify any horizontal and vertical asymptotes, and (c) verify your answer to part (a) by using a graphing utility and by creating a table of values.

1. \( f(x) = \frac{1 - 5x}{1 + 2x} \)
2. \[ f(x) = \frac{3x^2 + x - 5}{x^2 + 1} \]
In a pilot project, a rural township is given recycling bins for separating and storing recyclable products. The cost $C$ (in dollars) for supplying bins to $p\%$ of the population is given by

$$C = \frac{25,000p}{100-p}, \quad 0 \leq p < 100.$$ 

(a) Find the cost of supplying bins to 15% of the population.

(b) Find the cost of supplying bins to 50% of the population.

(c) Find the cost of supplying bins to 90% of the population.
(d) Use a graphing utility to graph the cost function. Be sure to choose an appropriate viewing window. Explain why you chose the values that you used in your viewing window.

(e) According to this model, would it be possible to supply bins to 100% of the residents? Explain.
Example: \( f(x) = \frac{x + 1}{x} \).

- **y-Intercept:** None
- **x-Intercept:** \((-1, 0)\)
- **Vertical asymptote:** \(x = 0\)
- **Horizontal asymptote:** \(y = 1\)
- **Additional points:** \((-2, 0.5), (-1.5, 1/3), (1, 2)\)
\[ g(x) = \frac{x - 2}{x^2 - 2x - 8} \]

**y-Intercept:**  
(0, -0.25)

**x-Intercept:**  
(2, 0)

**Vertical asymptote:**  
\( x = -2 \) and \( x = 4 \)

**Horizontal asymptote:**  
\( y = 0 \)

**Additional points:**  
(-4, -0.375), (0, 1/4), (6, 1/4)
\[ h(x) = \frac{x}{x^2 + 1} \]

**y-Intercept:** (0, 0)

**x-Intercept:** (0, 0)

**Vertical asymptote:** none

**Horizontal asymptote:** \( y = 0 \)

**Additional points:** (-2, -0.4), (-1, -1/2), (1, 1/2)
Examples: Sketch the graph of the rational function by hand. Use a graphing utility to verify your graph.

1. \[ g(x) = \frac{x}{x^2 - 9} \]

- **y-intercept**: \[ g(0) = 0 \] Point: \( (0, 0) \)
- **x-intercepts**: set \( x = 0 \) Point: \( (0, 0) \)
- **vert. asymptotes**: set \( x^2 - 9 = 0 \) \( x = \pm 3 \)
- **horiz. Asymptote**: let \( x \to \pm \infty \) \( y = 0 \)
2. \[ f(x) = \frac{x + 4}{x^2 + x - 6} \]
3. \[ g(x) = \frac{3x^4 - 5x + 3}{x^4 + 1} \]
**Definition:** If the degree of the numerator is exactly one more than the degree of the denominator, the graph of the function has a slant or oblique asymptote. To find the equation of the slant asymptote, divide the denominator into the numerator.
Definition: If the degree of the numerator is exactly one more than the degree of the denominator, the graph of the function has a slant or oblique asymptote. To find the equation of the slant asymptote, divide the denominator into the numerator.

If $n = m + 1$, then the graph of $f$ has a slant asymptote at $y = q(x)$, where $q(x)$ is the quotient from the division algorithm.
Example Sketch the graph of $y = \frac{x^2}{x - 2} = x + 2 + \frac{4}{x - 2}$

y-Intercept: (0, 0)

x-Intercept: (0, 0)

Vertical asymptote: $x = 2$

Slant asymptote: $y = x + 2$

Additional points: (-1/2, -0.1), (1, -1), (3, 9)
4. \[ f(x) = \frac{x^2 + 5x + 8}{x + 3} \]
The cost of producing \( x \) units is \( C = 0.25x^2 + 5x + 78 \). The average cost per unit is

\[
\overline{C} = \frac{0.25x^2 + 5x + 78}{x} = 0.25x + 5 + \frac{78}{x}.
\]

Find the number of units that should be produced to minimize the average cost. Graph this function on a graphing utility, then use the “minimum” command. \( x \approx 17.66 \)