Jigsaw

3.3 Real Zeros of Polynomial Functions
Synthetic Division
Remainder and Factor Theorem
Rational Zero Test
3.4 Fundamental Theorem of Algebra
Linear Factorization Theorem
Conjugate Pairs
Factoring a Polynomial

**Quiz** on End Behavior and Completing the Square

Homework: 3.3 & 3.4
\[ f(x) = a(x - h)^2 + k \]

\[ y = f(x) = a(x - 3)^2 - 5 \]

\[ 13 = a(0 - 3)^2 - 5 \]

\[ 13 = 9a - 5 \]

\[ 18 = 9a \]

\[ 2 = a \]
12. \( y = 2x^2 - 12x + 13 \)

\[
f(x) = a(x-h)^2 + k
\]

\[
f'(x) = 2x^2 - 12x + 13
\]

\[
f'(x) = 2 \left( x^2 - 6x + 9 \right) + 13 - 18
\]

\[
= 2 \left( x-3 \right)^2 - 5
\]

\[
\frac{1}{a}(6) = -3
\]

\[
(-3)^2 = 9
\]
12. $y = 2x^2 - 12x + 13$

23. $y = 2(x - 3)^2 - 5$
<table>
<thead>
<tr>
<th></th>
<th>13. $y = -2x^2 + 12x - 13$</th>
<th>24. $y = -2(x - 3)^2 + 5$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td><img src="image" alt="Graph" /></td>
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</tbody>
</table>
14. \[ y = -0.5x^2 + 3x + 0.5 \]
25. \[ y = -0.5(x - 3)^2 + 5 \]
15. \( y = -0.5x^2 - 3x - 9.5 \)  
26. \( y = -0.5(x + 3)^2 - 5 \)
16. \( y = 0.25x^2 + 1.5x - 2.75 \)

27. \( y = 0.25(x + 3)^2 - 5 \)
17. \( y = 4x^2 + 24x + 31 \)

28. \( y = 4(x + 3)^2 - 5 \)
7. \( x = -2 \), \( y = -3 \)

18. \( y = x^2 + 4x + 1 \)

29. \( y = (x + 2)^2 - 3 \)
19. \( y = -x^2 - 4x - 7 \)  
30. \( y = -(x + 2)^2 - 3 \)
20. \( y = -0.25x^2 - x + 2 \)

31. \( y = -0.25(x + 2)^2 + 3 \)
21. $y = -3x^2 + 12x - 9$

32. $y = -3(x - 2)^2 + 3$
<table>
<thead>
<tr>
<th>11.</th>
<th>22. $y = 0.5x^2 - 2x + 5$</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>33. $y = 0.5(x - 2)^2 + 3$</td>
</tr>
</tbody>
</table>
Divide $2x^3 - 5x^2 + x - 8$ by $x - 3$.

\[
\begin{array}{c|cccc}
3 & 2 & -5 & 1 & -8 \\
\hline
& 3 & 6 & 12 & 24 \\
\end{array}
\]

$q = \frac{13}{2} = 6 + R1$

$13 = (2)(6) + 1$

\[2x^3 - 5x^2 + x - 8 = (x-3)(2x^2 + x + 4) + 4\]

\[
2x^3 + x^2 + 4x \\
-6x^2 - 3x - 12 \\
\hline
2x^3 - 5x^2 + x - 8
\]
\[
\begin{array}{c}
x - 3 \overbrace{2x^3 - 5x^2 + x - 8} \\
2x^3 - 6x^2 \\
\overbrace{x^2 + x} \\
x^2 - 3x \\
4x - 8 \\
\underline{4x - 12}
\end{array}
\]
The result is \( 2x^2 + x + 4 + \frac{4}{x - 3} \).
The Division Algorithm

If \( f(x) \) and \( d(x) \) are polynomials such that \( d(x) \neq 0 \), and the degree of \( d(x) \) is less than or equal to the degree of \( f(x) \), there exist unique polynomials such \( q(x) \) and \( r(x) \) such that

\[
f(x) = d(x)q(x) + r(x)
\]

where \( r(x) = 0 \) or the degree of \( r(x) \) is less than the degree of \( d(x) \). If the remainder \( r(x) \) is zero, \( d(x) \) divides evenly into \( f(x) \).

\[
\frac{f(x)}{d(x)} \text{ is improper.} \quad \frac{r(x)}{d(x)} \text{ is proper.}
\]

\[
f(3) = 2(3)^3 - 5(3)^2 + 3 - 8
\]

\[
= 2 \cdot 27 - 5 \cdot 9 + 3 - 8 = 4
\]
Example: Perform long division and write in the form
\[ f(x) = d(x)q(x) + r(x). \]

1. \[ \frac{x^5 + 7}{x^3 - 1} \]

\[
\begin{array}{c|ccccc}
 & x^2 & + & 2x^2 & + 7 \\
\hline
x^3 - 1 & x^5 & + & 0x^4 & - x^2 & + 7 \\
 & \underline{-x^5 + x^2} & & \underline{+ x^2} & & \underline{+ 7} \\
 & 0 & + & 2x^2 & + 7 \\
 & \underline{- 2x^2} & & \underline{+ 2} \\
 & 0 & + & 9 \\
\end{array}
\]

\[
\begin{array}{c|cc}
 & 2 & + 13 \\
\hline
6 & 13 & + 1 \\
\end{array}
\]

Quotient: \[ 2x^2 + \frac{9}{x^3 - 1} \]

Remainder: 1
**Synthetic Division** is a short-cut process for dividing a polynomial of any degree by a polynomial of the form $x - k$.

Example: Use synthetic division to divide.

\[
\frac{5x^3 + 18x^2 + 7x - 6}{x + 3}
\]

\[
\begin{array}{c|cccc}
-3 & 5 & 18 & 7 & -6 \\
\hline
 & 5 & -9 & 6 \\
\hline
 & 5 & 3 & -2 & 0
\end{array}
\]

Thus, $5x^3 + 18x^2 + 7x - 6 = (x + 3)(5x^2 + 3x - 2)$ and $f(x) = (x + 3)(5x - 2)(x + 1)$.
The Remainder Theorem
If a polynomial \( f(x) \) is divided by \( x - k \), the remainder is \( r = f(k) \).

The Factor Theorem
A polynomial \( f(x) \) has a factor \( (x - k) \) if and only if \( f(k) = 0 \).
Using the Remainder in Synthetic Division

In summary, the remainder \( r \), obtained in synthetic division of \( f(x) \) by \( x - k \), provides the following information.

1. The remainder \( r \) gives the exact value of \( f \) at \( r = f(k) \).
2. If \( r = 0 \), \( (x - k) \) is a factor of \( f(x) \).
3. If \( r = 0 \), \( (k, 0) \) is an \( x \)-intercept of the graph of \( f \).
3. Use synthetic division and the Remainder Theorem to find \( f(6) = 1360 \) for \( f(x) = 10x^4 - 50x^3 - 800 \).
4. Show that \( x = -2 \) is a zero (or root) of \( x^3 + 2x^2 - 2x - 4 \). Factor completely and find all real zeros.

\[ x^3 + 2x^2 - 2x - 4 = (x+2)(x^2 - 2) = (x+2)(x-\sqrt{2})(x+\sqrt{2}) \]

\[ \pm 1 \pm 2 \pm 4 \]

\[ \begin{array}{cccc}
1 & 2 & -2 & -4 \\
\hline
1 & 3 & 1 & -3 \\
\hline
1 & 3 & 1 & -3 \\
\end{array} \]

\[ x^2 - 2 = (x-\sqrt{2})(x+\sqrt{2}) \]

\[ x^2 - 2 = 0 \]
\[ x^2 = 2 \]
\[ x = \pm \sqrt{2} \]
5. Use the Zero feature of your calculator to approximate the zeros of \( f(s) = s^3 - 12s^2 + 40s - 24 \) to three decimal places. Determine one of the exact zeros and use synthetic division to verify it. Factor completely.
The Rational Zero Test

If the polynomial $f(x) = a_n x^n + \ldots + a_1 x + a_0$ has integer coefficients, every rational zero of $f$ has the form

$$\text{Rational zero} = \frac{p}{q}$$

Where $p$ and $q$ have no common factors other than 1, $p$ is a factor of the constant term $a_0$ and $q$ is a factor of the leading coefficient $a_n$. 
Use the rational root theorem to solve:
Solve \( x^3 - 7x - 6 = 0 \).
Use the rational root theorem to solve:
Solve \( x^3 - 7x - 6 = 0 \).

\[ p: \pm 1, \pm 2, \pm 3, \pm 6 \]
\[ q: \pm 1 \]
\[ p/q: \pm 1, \pm 2, \pm 3, \pm 6 \]
Examples: List all possible rational zeros.

6. \( f(x) = 4x^4 - 17x^2 + 4 \)

\[
\begin{array}{cccccc}
6 & -1 & -13 & 8 \\
6 & 5 & -8 \\
\hline
6 & 5 & -8 & \mathbf{0}
\end{array}
\]

7. \( f(x) = 6x^3 - x^2 - 13x + 8 \)

\[
(x-1)(6x^2 + 5x - 8)
\]

\[
\pm 1 \pm 2 \pm 4 \pm 8
\]

\[
\pm 1 \pm 2 \pm 3 \pm 6
\]

\[
\pm 1 \pm \frac{1}{2} \pm \frac{2}{3} \pm \frac{4}{3} \pm \frac{8}{3}
\]

\[
\pm \frac{1}{6}
\]
Examples: Find all real zeros.

8. \( h(x) = -x^3 - 9x^2 + 20x - 12 \)
9. \[ f(z) = 12z^3 - 4z^2 - 27z + 9 \]
Synthetic Division (of a Cubic Polynomial)

To divide $ax^3 + bx^2 + cx + d$ by $x - k$, use the following pattern.

- **Vertical pattern:** Add terms.
- **Diagonal pattern:** Multiply by $k$.

Coefficients of dividend:
- $ka$

Coefficients of quotient:
- $a$
- $c$
- $d$

Remainder:
- $r$
The Fundamental Theorem of Algebra

If $f(x)$ is a polynomial of degree $n$, where $n>0$, then $f$ has at least one zero in the complex number system.
Linear Factorization Theorem

If \( f(x) \) is a polynomial of degree \( n \), where \( n > 0 \), \( f \) has precisely \( n \) linear factors

\[
f(x) = a_n(x - c_1)(x - c_2)\ldots(x - c_n)
\]

where \( c_1, c_2, \ldots, c_n \) are complex numbers. (Zeros may not be distinct!)
Solve $x^3 + 6x - 7 = 0$. 
\[ p: \pm 1, \pm 7 \]
\[ q: \pm 1 \]
\[ p/q: \pm 1, \pm 7 \]

Use synthetic division to find a number from the list that is a solution.

\begin{array}{c|cccc}
1 & 1 & 0 & 6 & -7 \\
 & 1 & 1 & 7 & 0 \\
\end{array}

\begin{array}{c|cccc}
1 & 1 & 7 & 0 \\
\end{array}
We know have \((x - 1)(x^2 + x + 7) = 0\). \(x - 1 = 0 \Rightarrow x = 1\).

\[x^2 + x + 7 = 0 \Rightarrow x = -\frac{1}{2} \pm \frac{3\sqrt{3}}{2}i.\]
Examples: Find all zeros and write as the product of linear factors.

1. \( f(y) = y^4 - 625 \)

2. \( f(x) = x^3 + 11x^2 + 39x + 29 \)
3. \( h(x) = x^4 + 6x^3 + 10x^2 + 6x + 9 \)
Complex Zeros Occur in Conjugate Pairs

Let $f(x)$ be a polynomial function that has real coefficients.

If $a + bi$, where $b \neq 0$, is a zero of the function, the conjugate $a - bi$ is also a zero of the function.
Factors of a Polynomial

Every polynomial of degree \( n > 0 \) with real coefficients can be written as the product of linear and quadratic factors with real coefficients, where the quadratic factors have no real zeros.

Definition: A quadratic factor with no real zeros is said to be prime or irreducible over the reals.
Factor \( f(x) = x^4 - 12x^2 - 13 \)

a) as the product of factors that irreducible over the rationals.

\[(x^2 - 13)(x^2 + 1)\]
b) as the product of factors that are irreducible over the reals.

\[(x + \sqrt{13})(x - \sqrt{13})(x^2 + 1)\]
c) completely.

\((x + \sqrt{13})(x - \sqrt{13})(x + i)(x - i)\)
Example: Write the polynomial \( f(x) = x^4 - 2x^3 - 3x^2 + 12x - 18 \)

(a) as the product of factors that are irreducible over the \textit{rationals},

(b) as the product of linear and quadratic factors that are irreducible over the \textit{reals},

(c) in completely factored form.