Mat 161 Agenda Day 23  3/23/05

3.2 Polynomials of Higher Degree
Zeros of Polynomial Functions
Intermediate Value Theorem
3.3 Real Zeros of Polynomial Functions
Long Division
Synthetic Division
Remainder and Factor Theorem
Rational Zero Test

Worksheet on Division

Homework:  3.2 & 3.3

Quiz on Friday:
Leading coefficient
Completing the square
P2 #13

\[ y = f(x) = 25 - x^2 = 0 \]

Vertex \((0, 25)\)

\[(5 - x)(5 + x) = 0\]

5 - x = 0 \(\Rightarrow\) 5 + x = 0

5 = x \(\quad\quad\quad\) x = -5
\[ f(x) = a(x-h)^2 + k \]

Standard form:

\[ -1(x-0)^2 + 25 \]

\[ f'(x) = 25 - x^2 \]
\( f (x) = 4x^2 - 4x + 21 \)

\( f (x) = 4\left(x^2 - x + \frac{1}{4}\right) + 21 - 1 \)

\( f (x) = 4\left(x - \frac{1}{2}\right)^2 + 20 = 0 \)

Vertex: \( \left(\frac{1}{2}, 20\right) \)

\[ \frac{1}{4}(-1) = -\frac{1}{2} \]

\[ (-\frac{1}{2})^2 = \frac{1}{4} \]

\( 4\left(x - \frac{1}{2}\right)^2 = -20 \)

\[ (x - \frac{1}{2}) = -5 \]

\[ \sqrt{(x - \frac{1}{2})^2} = \sqrt{5i^2} \]

\( \chi = \frac{1}{2} \pm i\sqrt{5} \)

\( \chi = \frac{1}{2} + i\sqrt{5} \)
\[ f(x) = -x^2 + 2x + 5 \]

\[ f(x) = -1(x^2 - 2x + 1) + 5 + 1 \]

\[ f(x) = -1(x-1)^2 + 6 = 0 \]

**Vertex:** \((1, 6)\)

\[ \frac{1}{2}(-2) = -1 \]

\[ (-1)^2 = 1 \]

\[-1(x-1)^2 = -6 \]

\[ (x-1)^2 = 6 \]

\[ x-1 = \pm \sqrt{6} \]

\[ x = \begin{cases} 1 + \sqrt{6} \\ 1 - \sqrt{6} \end{cases} \]

\( (1 + \sqrt{6}, 0) \), \( (1 - \sqrt{6}, 0) \)
Review of Leading Coefficient Test

Examples: Use the Leading Coefficient Test to determine the right-hand and left-hand behavior of the graph of the polynomial function. Use a graphing utility to verify your result.

1. \( f(x) = \frac{1}{3}x^3 + 5x \)

2. \( f(x) = \frac{3x^4 - 2x + 5}{4} \)

\[
\begin{align*}
  x & \to -\infty & y & \to -\infty \\
  x & \to +\infty & y & \to +\infty \\
\end{align*}
\]
3. \[ f(s) = -\frac{7}{8}(s^3 + 5s^2 - 7s + 1) \]
Real Zeros of Polynomial Functions

If \( f \) is a polynomial function and \( a \) is a real number, the following statements are equivalent.

1. \( x = a \) is a zero of the function \( f \).
2. \( x = a \) is a solution of the polynomial equations \( f(x) = 0 \).
3. \( (x - a) \) is a factor of the polynomial \( f(x) \).
4. \( (a,0) \) is an \( x \)-intercept of the graph of \( f \).
Repeated Zeros

For a polynomial function, a factor of \((x-a)^k, k > 1\), yields a **repeated zero** \(x = a\) of multiplicity \(k\).

1. If \(k\) is odd, the graph **crosses** the \(x\)-axis at \(x = a\).
2. If \(k\) is even, the graph **touches** (but does not cross) the \(x\)-axis at \(x = a\).
Example:

\[f(x) = x^3 - x^2 - x + 1.\]
The x-intercepts are (-1, 0) and (1, 0).
Sketch the graph of $f(x) = x^3 - 2x^2$.
1. Since $f(x) = x^2(x - 2)$, the x-intercepts are (0, 0) and (2, 0). Also, 0 has multiplicity 2, therefore the graph will just touch at (0, 0).
2. The graph will rise to the right and fall to the left.
3. Additional points on the graph are (1, -1), (-1, -3), and (-2, -16).
Examples: Find all the real zeros of the polynomial functions. Determine the multiplicity of each zero. Use a graphing utility to verify your result.

4. \( f(x) = 49 - x^2 \)

5. \( g(x) = 5(x^2 - 2x - 1) \)
\[ y = t \left( 4t^2 - 6t + 9 \right) \]

\[ g(t) = t(t^2 - 3)(t^2 - 3) \]

\[ t^2 - 3 = 0 \]
\[ t^2 = 3 \]
\[ t = \pm \sqrt{3} \]
\[ t \left( t^2 - 3 \right) \left( t^2 - 3 \right) \]
\[ t \left( t + \sqrt{3} \right) \left( t - \sqrt{3} \right) \left( t + \sqrt{3} \right) \left( t - \sqrt{3} \right) \]
\[ t \left( t + \sqrt{3} \right)^2 \left( t - \sqrt{3} \right)^2 \]
8. \[ h(x) = \frac{1}{3} x^3 (x - 4)^2 \]

9. \[ f(x) = \frac{1}{5} (x + 2)^2 (3x - 5)^2 \]
Example: Find a polynomial with the given zeros.

10. 0, 2, 7

\[ x = 0 \quad \Rightarrow \quad (x-0) \]
\[ x = 2 \quad \Rightarrow \quad (x-2) \]
\[ x = 7 \quad \Rightarrow \quad (x-7) \]

\[ f(x) = x \cdot (x-2) \cdot (x-7) \]
\[ f(x) = x^3 - 9x^2 + 14x \]
Intermediate Value Theorem (IVT)

Let $a$ and $b$ be real numbers such that $a < b$. If $f$ is a polynomial function such that $f(a) \neq f(b)$, then in the interval $[a, b]$, $f$ takes on every value between $f(a)$ and $f(b)$. 
Use the Intermediate Value Theorem to approximate the real zero of $f(x) = 21x + 18$ on $[0, 1]$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>18</td>
</tr>
<tr>
<td>1</td>
<td>-6</td>
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</tbody>
</table>

$f(0) = 18$ and $f(1) = -6$. By the Intermediate Value Theorem, there exists at least one real zero between 0 and 1.
Note that $f(0) = 18$ and $f(1) = -6$. Therefore, by the Intermediate Value Theorem, there must be a zero between 0 and 1.

Note further that $f(0.7) = 1.242$ and $f(0.8) = -1.232$. Therefore there must be a zero between 0.7 and 0.8.
Example: Use the IVT to find intervals of length 1 in which $f$ is guaranteed to have a zero. Find the zeros.

(Hint: Use the TABLE function on your calculator.)

11. $h(x) = x^4 - 10x^2 + 2$
Mat 161 Worksheet Division

Do the division in the following two problems without using a calculator, write down all the steps of the process.

1. \[
\begin{array}{c}
289 \\
\hline
25 \\
\hline
224 \\
\hline
280 \\
\hline
243 \\
\hline
225 \\
\hline
18
\end{array}
\]

Rm 18
2. \( \frac{6x^3 - 19x^2 + 16x - 4}{x - 2} \)

\[ \begin{array}{c}
2 \quad \frac{7}{2} = 3 + R1 \\
6x^2 - 7x + 2 \\
7 = 2 \cdot 3 + 1 \\
\end{array} \]

\[ \begin{array}{c}
\text{\(x - 2\)} \quad | \quad 6x^3 - 19x^2 + 16x - 4 \\
\text{\(3\)} \\
\text{\(-7x^2 + 16x\)} \\
\text{\(-7x^2 + 14x\)} \\
\text{\(2x - 4\)} \\
\text{\(-2x - 4\)} \\
\end{array} \]
\[
\frac{6x^3 - 16x^2 + 17x - 6}{3x - 2}
\]
Divide $2x^3 - 5x^2 + x - 8$ by $x - 3$. 
The result is $2x^2 + x + 4 + \frac{4}{x-3}$.
The Division Algorithm

If \( f(x) \) and \( d(x) \) are polynomials such that \( d(x) \neq 0 \), and the degree of \( d(x) \) is less than or equal to the degree of \( f(x) \), there exist unique polynomials such \( q(x) \) and \( r(x) \) such that

\[
f(x) = d(x)q(x) + r(x)
\]

where \( r(x) = 0 \) or the degree of \( r(x) \) is less than the degree of \( d(x) \). If the remainder \( r(x) \) is zero, \( d(x) \) divides evenly into \( f(x) \).

\[
\frac{f(x)}{d(x)} \text{ is improper.} \quad \frac{r(x)}{d(x)} \text{ is proper.}
\]
Example: Perform long division and write in the form 
\[ f(x) = d(x)q(x) + r(x). \]

1. \[ \frac{x^5 + 7}{x^3 - 1} \]
**Synthetic Division** is a short-cut process for dividing a polynomial of any degree by a polynomial of the form $x - k$.

Example: Use synthetic division to divide.

2. $\frac{5x^3 + 18x^2 + 7x - 6}{x + 3}$
The Remainder Theorem
If a polynomial $f(x)$ is divided by $x - k$, the remainder is $r = f(k)$.

The Factor Theorem
A polynomial $f(x)$ has a factor $(x - k)$ if and only if $f(k) = 0$. 
Using the Remainder in Synthetic Division

In summary, the remainder \( r \), obtained in synthetic division of \( f(x) \) by \( x - k \), provides the following information.

1. The remainder \( r \) gives the exact value of \( f \) at \( r = f(k) \).
2. If \( r = 0, (x - k) \) is a factor of \( f(x) \).
3. If \( r = 0, (k, 0) \) is an \( x \)-intercept of the graph of \( f \).
3. Use synthetic division and the Remainder Theorem to find \( f(6) \) for \( f(x) = 10x^4 - 50x^3 - 800 \).
4. Show that \( x = -2 \) is a zero (or root) of \( x^3 + 2x^2 - 2x - 4 \). Factor completely and find all real zeros.
5. Use the Zero feature of your calculator to approximate the zeros of \( f(s) = s^3 - 12s^2 + 40s - 24 \) to three decimal places. Determine one of the exact zeros and use synthetic division to verify it. Factor completely.
The Rational Zero Test

If the polynomial \( f(x) = a_n x^n + \ldots + a_1 x + a_0 \) has integer coefficients, every rational zero of \( f \) has the form

\[
\text{Rational zero} = \frac{p}{q}
\]

Where \( p \) and \( q \) have no common factors other than 1, \( p \) is a factor of the constant term \( a_0 \) and \( q \) is a factor of the leading coefficient \( a_n \).
Use the rational root theorem to solve:
Solve $x^3 - 7x - 6 = 0$. 
Use the rational root theorem to solve:
Solve $x^3 - 7x - 6 = 0$.

$p$: $\pm 1, \pm 2, \pm 3, \pm 6$

$q$: $\pm 1$

$p/q$: $\pm 1, \pm 2, \pm 3, \pm 6$
Examples: List all possible rational zeros.

6. \( f(x) = 4x^4 - 17x^2 + 4 \)

7. \( f(x) = 6x^3 - x^2 - 13x + 8 \)
Examples: Find all real zeros.

8. \( h(x) = -x^3 - 9x^2 + 20x - 12 \)
9. \( f(z) = 12z^3 - 4z^2 - 27z + 9 \)
Synthetic Division (of a Cubic Polynomial)

To divide $ax^3 + bx^2 + cx + d$ by $x - k$, use the following pattern.

- **Vertical pattern:** Add terms.
- **Diagonal pattern:** Multiply by $k$.

<table>
<thead>
<tr>
<th>k</th>
<th>a</th>
<th>b</th>
<th>c</th>
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Coefficients of dividends

Coefficients of quotient

Remainder